A Multi-angle Research on the Solution of Function Test Questions in College Entrance Examination

Fangfang Chen[1*], Dan Wan[2]
[1]*School of mathematics, Sichuan University of Arts and Science, Dazhou, Sichuan, China.*
[2]*Sanli Primary School of Dazhou District, Dazhou, Sichuan, China.*
**Corresponding Author.*

Abstract: Functions are a fundamental component of high school mathematics and pose a challenge for students. The college entrance examination questions prioritize the cultivation of moral character and abilities, which is consistent with the curriculum standards. They evaluate students' basic knowledge, skills, and subject core literacy, and are also a means of talent selection. By analyzing a representative question (Question 21 from the 2022 National College Entrance Examination Mathematics Subject for Science), this study examines solution methods and uncovers underlying thoughts and approaches within the question. It emphasizes both general problem-solving principles and methods as well as problem-solving within advanced mathematical thinking. Drawing from this research, targeted teaching strategies for problem-solving are proposed. Through analysis of representative questions, this paper aims to chart an effective path for high school mathematics education while providing strategies and recommendations for review and preparation.

Keywords: Function; Derivative; Constructor Function; Core Quality

1. Introduction

The college entrance examination, as a unified national examination, aims to select outstanding talents suitable for the stage of higher education through testing the knowledge and ability of candidates, and assign them to different majors and subject areas. However, as a basic subject, mathematics is not only the abstract generalization and strict description of human beings to space things. In the development of human history and social life, mathematics plays an irreplaceable role, and is also an essential basic tool for learning and researching modern science and technology. Therefore, the annual math high test questions have been widely concerned by people, and people continue to analyze and answer them. But what should we focus on? How to analyze and answer questions in order to obtain the exact enlightenment for mathematics teaching and learning.

Function is one of the most basic concepts in modern mathematics, and it is a mathematical language and tool to describe the relationship and change law of variables in the objective world [1]. The derivative is the most powerful tool to study variables, using the derivative to study some properties of functions and curves, so as to solve some practical problems. In the mathematics questions of the college entrance examination, derivative questions based on function knowledge often appear as the final question, which is also the part that most students are most likely to lose points. It is not difficult to find that the usual method of asking derivative questions revolves around the following aspects, such as proving inequality, discussing monotonicity, finding the maximum value, finding the extreme value and finding the range of parameter values. Function problems are highly comprehensive, involving not only the understanding and mastery of the general method, but also the flexibility and innovation of thinking, and a relatively comprehensive examination of students' essential knowledge, key abilities and core mathematical literacy [2]. The general (1) problem is relatively simple, using conventional ideas and solutions, it is easy to get the answer; However, the question (2) is more difficult and the calculation is more complex, and some questions are even propositions based on the knowledge of higher mathematics. If teachers can stand in a higher perspective and use the relevant knowledge of higher mathematics to analyze, they can deeply understand the meaning of the question and effectively grasp the key information of the question.

Therefore, function and derivative become the focus of college entrance examination research. The exploration of its solution is of great significance for inspiring teachers' teaching and providing certain teaching guidance for education and teaching. The 2022 college entrance examination math questions focus on the examination of "four foundations" and "four abilities" as well as core mathematical literacy, and strive to reflect the essence of mathematical content, requiring students to pay more attention to the connection and comprehensiveness of mathematical content when solving problems. Attach importance to the application of model and the application of mathematical ideas (such as the combination of number and form, algorithm, transformation, equation and meta-thought) in the process of solving problems [3]. In this study, taking the National Grade A Paper of Science and Mathematics No.21 title in 2022 as an example, in-depth analysis of the intention of the exam questions, multi-dimensional perspective solution, to find the correct path for the teaching of high school mathematics, and provide strategies and suggestions for the review and preparation of the exam.

2. Exam Paper (Question 21 from the 2022 National College Entrance Examination Mathematics Subject for Science)

Given the function

$$f(x)=\frac{e^x}{x}-\ln x+x-a \quad (1)$$

(1) If $f(x)>0$, find the range of values of a.

(2) Proof: if $f(x)$ has two zero points x_1, x_2, then $x_1x_2<1$.

This problem belongs to the typical extreme point deviation problem. The monotone and extremum of the function are studied by means of the derivative, and the range of the parameter is found under the condition of given constancy (that is, known minimum value).Then the relationship between the two zero points is studied on this basis [4-5].

The first question is relatively simple, mainly examining the basic knowledge of derivative, monotonicity, extreme value (maximum value) and the basic skills of parameter-variable separation and construction function; Question(2) is more difficult, requiring candidates to master the relevant nature of the function and the basic methods and tools of research, and be able to flexibly apply. From the aspect of core literacy, it is difficult to change the test questions from situation (A hybrid operation involving exponents and logarithms) to problem (proof of inequality), which tests the ability of students to flexibly apply functions, equations and inequalities to solve complex problems, and has higher requirements for core literacy such as mathematical operation, mathematical abstraction and logical reasoning. It has good comprehensive training value and teaching inquiry value. The examination of mathematical thought and method mainly reflects the ideas of number combination and transformation.

3. Multi-dimensional View Solution

For the question (1), there are two main ways to solve the problem.

Approach 1: Directly compute the derivative to get the range of parameters.

$$\begin{aligned} f'(x) &= \frac{e^x x-e^x}{x^2}-\frac{1}{x}+1 \\ &= \frac{(e^x+x)(x-1)}{x^2},\ x\in(0,+\infty) \end{aligned} \quad (2)$$

Let $f'(x)=0$, we can get $x=1$. Next we discuss the properties of the function by category.

The function $f(x)$ is said to be monotonically decreasing if $f'(x)<0$, for $x\in(0,1)$. On the other hand, we have $f'(x)>0$ for $x\in(1,+\infty)$. And the function $f(x)$ is creasing. It follows that we can get the minimum value of $f(x)$, namely $f_{\min}(x)=f(1)=e+1-a$. Since $f(x)\geq 0$. It is easy to get the range of values of a, namely $a\in(-\infty,\ e+1]$.

Analysis: This question belongs to a basic type of problem, which tests the basic operations and properties of derivatives. After taking the derivative, the fraction needs to be simplified by performing long division and factoring the numerator. This requires a high level of mathematical computation literacy. In daily teaching, teachers should not only emphasize the explanation and explanation of calculation rules to avoid students' confusion or forgetfulness of formulas [6]. They should also be good at guiding students to summarize problem-solving methods, improve their mathematical computation ability, promote mathematical thinking development, and cultivate the habit of thinking deeply and summarizing.

Approach 2: Separation of Parameters and Construction of Functions.

Since $f(x) \geq 0$, we can easy to get the inequality $a \leq \frac{e^x}{x} - \ln x + x$.

Then let $h(x) = \frac{e^x}{x} - \ln x + x$, and the derivative is used to obtain the minimum value of this function $h(x)$.

Analysis: Parametric-variable separation is one of the commonly used methods to find the range of parameter values. This method can avoid the trouble of classification discussion. It is often used to solve the invariability of inequality, the solution of inequality, the zero of function, and the range of parameter values in the monotonicity of function. In this problem, the inequality $f(x) \geq 0$ can be transformed, $a \leq \frac{e^x}{x} - \ln x + x$ and then a new function is constructed to solve the problem.

Now we discuss the second question. There are three main ways to solve it.

Idea 1: Multi-wheel construction, twice derivation.

According to the conclusion of question (1), the condition that the function $f(x)$ has two zero points is $a \leq e+1$, and one in each interval (0,1) and $(1, +\infty)$.

Assuming that $0 < x_1 < 1 < x_2$, where: $\frac{1}{x_2} \in (0, 1)$. And the function is decreasing on (0,1), it is easy to translate the conclusions to be proved into the following formula

$$x_1 x_2 < 1 \Leftrightarrow f(x_2) > f(\frac{1}{x_2}). \tag{3}$$

Namely,

$$\frac{e^{x_2}}{x_2} - \ln x_2 + x_2 > x_2 e^{\frac{1}{x_2}} + \ln x_2 + \frac{1}{x_2} \tag{4}$$

For the proof of formula (4), the main idea is to construct an auxiliary function, and then use the derivative to judge the monotonicity of the function and solve the problem. However, there are various ways to construct auxiliary functions, on the one hand, it can be constructed directly, on the other hand, it can also be transformed and then constructed. Such as let:

$$\mathrm{g(x)} = \frac{e^x}{x} - xe^{\frac{1}{x}} - 2lnx + x - \frac{1}{x} \quad \text{for}(x > 1) \tag{5}$$

Or firstly, equation (4) can been transformed the following inequality

$$\frac{e^{x_2}}{x_2} + \ln \frac{e^{x_2}}{x_2} > x_2 e^{\frac{1}{x_2}} + \ln (x_2 e^{\frac{1}{x_2}}) \tag{6}$$

now let $s(x) = x + \ln x$, and then just prove the inequality $\mathrm{s}(\frac{e^{x2}}{x_2}) > \mathrm{s}(x_2 e^{\frac{1}{x2}})$, the procedure is not described in detail here.

Analysis: The most common way to solve the extreme point deviation problem is to directly construct the symmetric function, which is difficult for students to break through. Therefore, the idea in this research is to start with the conclusion and reverse analyze the problem of transforming bi-variate into variate. Namely

$$x_1 x_2 < 1 \Leftrightarrow f(x_2) > f(\frac{1}{x_2}) \tag{7}$$

Utilize general methodologies and techniques for iterative function construction, as well as problem-solving through derivative applications. The process of proving complex functions $g'(x) > 0$ may involve initial simplification of intricate factors through scaling and transformation before computation, which is a commonly employed technique in problems solving. Based on the above solution methods, it is found that the isomorphic function method has a unique effect on the analytical formula in which the reference function and the pair function appear simultaneously, such as the following typical functions $xe^x, x \ln x, \frac{\ln x}{x}$, which the principle of transformation is to use identities, and common identities are $x = e^{\ln x}$, $x = \ln e^x$. Students should be encouraged to bold attempt in teaching, reasonable variations, for a variety of ways to solve the problem [4].

Idea 2: Use logarithmic mean inequality wisely

Logarithmic mean inequality can be found in the formula (8)

$$\sqrt{x_1x_2} < \frac{x_1 - x_2}{\ln x_1 - \ln x_2} < \frac{x_1 + x_2}{2} \quad (8)$$

Assume $0 < x_1 < 1 < x_2$, from the equality $f(x_1) = f(x_2)$ follows that

$$e^{(x_1 - \ln x_1)} + (x_1 - \ln x_1) = e^{(x_2 - \ln x_2)} + x_2 - \ln x_2 \quad (9)$$

Simplify formula (9)

$$\frac{x_2 - x_1}{\ln x_2 - \ln x_1} = 1. \quad (10)$$

From (10) and the inequality $\sqrt{x_1x_2} < 1$ to be proved, we can get:

$$\sqrt{\frac{x_2}{x_1}} - \sqrt{\frac{x_1}{x_2}} > \ln\frac{x_2}{x_1} \quad (11)$$

Let $t = \sqrt{\frac{x_2}{x_1}} > 1$, the formula (11) will be transformed

$$g(t) = 2\ln t - t + \frac{1}{t} > 0 \quad (12)$$

And its derivative $g'(t) = -(1 - \frac{1}{t})^2$

therefore $g(t) < g(1) = 0$,and the proof of the inequality $\sqrt{x_1x_2} < 1$ is complete.

Analysis: The logarithmic mean inequality is a fundamental tool for solving inequalities, particularly in the context of comprehensive function problems encountered in college entrance examinations [7-8]. Approach 2 begins by establishing an equivalence relationship $f(x_1) = f(x_2) = 0$, and then use the log-mean inequality to solve the problem by isomorphism, substitution, and construction of new functions. It astutely utilizes the logarithmic mean inequality to resolve the problem. This approach ingeniously leverages the number 1 to establish a connection between two variables and surmounts challenging points, necessitating students to possess a strong foundation and adeptness in flexible application."

Idea 3: Ratio substitution, construct function.

Formula (9), we can also get:

$$\frac{e^{x_1}}{x_1} + \ln\frac{e^{x_1}}{x_1} = \frac{e^{x_2}}{x_2} + \ln\frac{e^{x_2}}{x_2}, \quad (13)$$

While the function $g(t) = t + \ln t$ is creasing for $t \in (0, +\infty)$ .Hence

$$\frac{x_2}{x_1} = e^{x_2 - x_1} \quad (14)$$

$$\text{Note } s = \frac{x_2}{x_1} > 1. \quad (15)$$

From (14) and (15) follow that

$$x_1x_2 = \frac{s\ln^2 s}{(s-1)^2} \quad (16)$$

Therefore the conclusion turn into

$$\frac{s\ln^2 s}{(s-1)^2} < 1 \quad (17)$$

The formula (15) can be transformed

$$\ln s < \frac{s-1}{\sqrt{s}} \quad (18)$$

Let $t = \sqrt{s}$, and the formula (17) can be transformed

$$p(t) = 2\ln t - t + \frac{1}{t} < 0 \quad (19)$$

Obviously, the formula (19) is easy to prove, and the conclusion is complete.

Analysis: the method of ratio substitution is also a common method, its purpose is to eliminate [9-10]. Of course, it has a wide range of applications, such as extreme point migration problems and the transformation of bi-variate inequalities into variate problems. However, in practical applications, it is flexible and changeable, requiring candidates to be flexible and flexible.

4. Conclusions

Function and its derivatives are an important part of mathematics teaching in high school, and also a difficult point for students to learn. This kind of questions can test students' ability of flexible use of knowledge, innovative thinking, logical thinking and operational literacy from a higher dimension. Therefore, the requirements for students are higher, not only to have a solid basic knowledge, but also to have the ability to flexibly use knowledge, in order to solve problems. However, students' knowledge of function is very scattered, and students are easy to make mistakes in the exam. Through sorting out the math questions of the 2022 college entrance examination, the characteristics of the questions are analyzed from the aspects of essential knowledge, key ability and discipline literacy. And from the whole point of view to analyze the test questions of the foundation, connection, comprehensiveness, application and innovation. As high school mathematics teachers, they should teach students effective problem-solving methods from a higher dimension, guide students to summarize, establish a complete knowledge system for students, and constantly improve students' problem-solving ability. Based on the above analysis of the subject, the following teaching enlightenment is obtained.

4.1 Establish a Solid Foundation for the Discipline based on Fundamental Concepts.

The mathematical concept is the foundation of the mathematical building, the basic element of the mathematical knowledge system, and the cell of mathematical thinking. Function is a very important concept in analysis, which runs through elementary mathematics and advanced mathematics. In learning and exam preparation, students must first have a deep understanding of the connotation of the concept. Teachers should explain the occurrence and development process of the basic concept, reveal the essence of the concept, establish the relationship between the concepts, guide students to analyze, solve and deepen the problem, and help students to understand. For example, function and derivative, function is to describe the relationship between variables, and derivative is the exact description of the concept of the rate of change of the function. The derivative itself is a function, but it is also a powerful tool to study the function, the two complement each other.

4.2 Emphasis on Basic Models and Emphasis on General Methods

The fundamental elementary functions commonly encountered in high school examinations include power functions, exponential functions, logarithmic functions, and trigonometric functions. Despite the consistency in knowledge content, the examination format is subject to continual change with flexible and diverse testing methodologies. Therefore, it is imperative for educators to guide students towards a comprehensive study of these prevalent function models, such as $y=xe^x$, $y=x/e^x$, $y=x\ln x$, $y=e^x/x$, $y=x/\ln x$, $y=e^x\pm x$, $y=\ln x\pm x$, e.c.

By summarizing and concluding the generality method of studying function, we can understand the idea that the carrier of function is changing but the research method is unchanged, and guide students to find the variance in the change, adapt to all changes with the variance, and realize multiple problems and one solution. Avoid question types and routines, mechanically apply them, encourage students to think actively when facing unfamiliar problem situations, make bold attempts, and actively explore.

4.3 Infiltrate Mathematical Thoughts and Pay Attention to Essential Exploration

Mathematical thought and method are contained in the process of the formation, development and application of mathematical knowledge, which is to reach a higher level of abstraction and generalization of mathematical knowledge and method. The examination of function and derivative in the college entrance examination focuses on understanding and application, and pays attention to the combination with mathematical thought and method. Transformation and classification discussion are important ways to solve function and derivative problems.

For example, the general solution to the extreme value deviation problem in (2) is to directly construct symmetric functions $g(x) = f(a/x) - f(x)$. If students do not understand the essence, they will not encounter similar problems the next time. The first idea in this paper is to get the unequal relationship between the two roots of the function from the conclusion analysis, combined with the general method of proving inequality, it is easy to think of solving the problem through the constructor function. Therefore, in teaching, we should pay attention to the refinement of mathematical thoughts and methods, guide students to understand the mathematical thoughts contained in the questions, let students understand why they do so, and truly learn to analyze and solve problems.

This study deeply analyzes the intention of the test questions and solutions from multi-dimensional perspectives, so as to find the correct path for the teaching of high school mathematics, provide strategies and suggestions for reviewing and preparing for the test, and hope to provide relevant teaching strategies for front-line teachers.

Acknowledgments

This work was supported by the Fundation of 2022 High-level Talent Project of Science and Technology Division of Sichuan University of Arts and Sciences (2022CC01Z); "Study on Middle School Students' Learning Participation Focusing on Core Literacy", Sichuan University of Arts and Sciences (2022QD026).

References

[1] Gengping Zeng, Study on "Function and derivative" test questions of the National College Entrance Examination in 2022. Research on Middle School Mathematics (South China Normal University Edition), 2023(03): 6-9.

[2] Yumei Xia, Yiling Lu, Lianhua Ning, "Deep Digging" of a function derivative question-Analysis and teaching Enlightenment of the 22nd question of Volume I of the new College Entrance Examination in 2022. Educational Research and Review (Secondary Education and Teaching), 2023(04): 34-39.

[3] Wenjing Li, Kun Zhang, Teaching design of college entrance Examination questions from the perspective of Higher Mathematics: A case study of the final topic of the new College Entrance Examination II in 2023. Research on Middle School Mathematics (South China Normal University Edition), 2024(02): 36-39.

[4] Changbo Liu, Analysis of extreme point deviation problem. Mathematics, Physics and Chemistry of Middle School students (Sophomore Mathematics, College Entrance Examination Mathematics), 2024(6): 28-30.

[5] Jun Zhang, Wuxue Li, Zeyu Hu, Application of the construction method to the extreme point migration problem: starting from the question 21 of the 2022 National College Entrance Examination. Middle School Mathematics Research, 2022(12): 43-45.

[6] Qiang LI, Application of isomorphic function and extreme point migration problem. Mathematics, Physics and Chemistry of Middle School Students (College Entrance Examination Mathematics), 2023(05): 33-36.

[7] Jigang Chen, Yu Long, Application of log-mean Inequality in College Entrance Examination and Competition. Research on Middle School Mathematics (South China Normal University Edition), 2019(10): 46-47.

[8] H Y Huang, Log-mean inequality and its application. Mathematical and Physicochemical Problem Research, 2022(31): 74-76.

[9] Hai-Lan LI, Six proofs and background analysis of an extreme point deviation problem. Mathematics Learning and Research, 2023(34): 116-118.

[10] Huiqing Guo, Wenhui Huang, Yiwei Ge, The red mountain is in full bloom, and the night is also red: Analysis of the mathematical questions of 2022 College Entrance Examination and Suggestions for preparation. Mathematics Education in China (Senior High School Edition), 2022(7): 3-20.