Comprehensive Dynamic Analysis of a Time-Delayed SIR Epidemiological Model

Quanhao Song
School of Sciences, Southwest Petroleum University Chengdu, Sichuan, China

Abstract: In this paper, the Hopf bifurcation of the time-delay epidemic model is deeply studied, and the stability and bifurcation conditions are analyzed by combining theory and numerical simulation, so as to provide a scientific basis for the prevention and control strategy. Firstly, the epidemic dynamics model with time delay is constructed. Then by applying stability theory and bifurcation theory, we have conducted a detailed analysis of the local stability of the model equilibrium point and identified the delay threshold *τ0* that causes Hopf bifurcation to occur. Using mathematical theory and numerical calculation, the time-delay dynamic figure of the system is drawn, which provides theoretical support for the prevention and control of infectious diseases. The analysis of complex mathematical phenomena has deepened our understanding of infectious disease prevention, enriched the delay model theory, and helped formulate scientific prevention and control strategies.

Keywords: Sir Model; Basic Reproductive Number; Time Delay; Hopf Bifurcation.

1. Introduction

When dealing with emerging threats, it is crucial to understand and accurately predict the dynamics of infectious disease transmission. These insights are not only the premise of taking effective prevention and control measures, but also the cornerstone of formulating public health policies and ensuring rapid and appropriate response to sudden outbreaks. The complex and dynamic nature of infectious diseases has stimulated the wide interest of the scientific community and promoted the development of diversified mathematical models. These models have deeply analyzed the trend of the epidemic situation, greatly enriched our cognition and enhanced the social response and preparation ability.

The classic SIR model was proposed by Cooke[1], which describes the spread of epidemics in a population through a compartment model. Subsequently, many scientists also extended SIR and proposed SEIR[2] and SEIQR[3].

Time delay, the lag effect in the transmission, is widespread in the transmission of infectious diseases, covering the latency from exposure to symptom appearance and the non-timeliness of the recovery process. Incorporating time lag factors into the model has immeasurable value for accurately describing the dynamic evolution of infectious diseases. It can significantly improve our understanding and prediction ability of epidemic development, and make the prediction results closer to reality.

Liu et.al[4] considered the dynamic behavior and optimal control problem of a class of time-delay model, and first discussed the influence of time-delay on stability, and also found that Hopf bifurcation occurs at a specific time delay.

Tchuenche and Nwagwo[5] constructed a class of SIR models with time delay based on the research results of multiple scholars. In order to further explore the stability characteristics of the model, they ingeniously applied the Lyapunov function method for analysis and effectively verified the reliability of the conclusions obtained through numerical simulation. Furthermore, we note that with subtle variations in time delay, the equilibrium point of the system may exhibit complex bifurcation phenomena. Based on this new discovery, we conducted an in-depth analysis of the existence of Hopf bifurcation, aiming to reveal the deeper impact of time delay on the dynamic behavior of infectious disease models.

2. Dynamic Analysis

Based on the achievements of many scholars, a class of time-delay SIR models is proposed:

$$\begin{cases} \dfrac{dS}{dt} = -\beta e^{-\mu_1 \tau} S(t) I(t-\tau) - \mu S(t) + \gamma \\ \dfrac{dI}{dt} = \beta e^{-\mu_1 \tau} S(t) I(t-\tau) - (\mu + \xi) I(t) \\ \dfrac{dR}{dt} = \xi I(t) - \mu R(t) \end{cases} \tag{1}$$

where, β is effective daily contact rate, μ_1 is disease-induced death rate in the interval $(0,\tau)$, μ is death rate of the population ($\mu_1 << \mu$), γ is birth rate, ξ is daily recovery rate of the infection, τ is time delay.
Using the regeneration matrix method, they calculate the basic reproduction number.
Since the first two equations are independent of the second equation, we simplify the above equation system to obtain:

$$R_0 = \frac{\beta e^{-\mu_1\tau}}{\mu+\xi} \tag{2}$$

$$\begin{cases} \dfrac{dS}{dt} = -\beta e^{-\mu_1\tau} S(t) I(t-\tau) - \mu S(t) + \gamma \\ \dfrac{dI}{dt} = \beta e^{-\mu_1\tau} S(t) I(t-\tau) - (\mu+\xi) I(t) \end{cases} \tag{3}$$

Hopf Bifurcation
The initial condition is $\phi = \{\phi_1, \phi_2\}$,
$C_+ = \{\phi \in C([-\tau, 0]), R_+^2\}$, where,

$$R_+^2 = \{(S,I) \in R^2 : S \ge 0, I \ge 0\}, \phi_i > 0, i = 1,2. \tag{4}$$

At this point, we further analyze the bifurcation and periodicity of the equilibrium point of endemic diseases.
Assume endemic equilibrium $E^* = (S^*, I^*)$, solve

$$\begin{cases} -\beta e^{-\mu_1\tau} S^* I^* - \mu S^* + \gamma = 0 \\ \beta e^{-\mu_1\tau} S^* I^* - (\mu+\xi) I^* = 0 \end{cases} \tag{5}$$

we get

$$S^* = \frac{\mu+\xi}{\beta} e^{\mu_1\tau}, I^* = \frac{\gamma}{\mu+\xi} - \frac{\mu}{\beta} e^{\mu_1\tau} \tag{6}$$

$$E^* = \left(\frac{\mu+\xi}{\beta} e^{\mu_1\tau}, \frac{\gamma}{\mu+\xi} - \frac{\mu}{\beta} e^{\mu_1\tau} \right) \tag{7}$$

We obtain endemic equilibrium.
Then, the characteristic equation of E^*

$$|\lambda E - J| = \begin{vmatrix} \lambda+\mu+\beta e^{-(\mu_1+\lambda)\tau} I^* & \beta e^{-(\mu_1+\lambda)\tau} S^* \\ -\beta e^{-(\mu_1+\lambda)\tau} I^* & \lambda+\mu+\xi-\beta e^{-(\mu_1+\lambda)\tau} S^* \end{vmatrix} = 0 \tag{8}$$

we can get

$$\lambda^2 + n_1\lambda + n_2 + e^{-(\mu_1+\lambda)\tau}(n_3\lambda + n_4) = 0 \tag{9}$$

where,
$n_1 = 2\mu+\xi, n_2 = \mu^2+\mu\xi, n_3 = \beta I^* - \beta S^*, n_4 = \beta I^*\mu - \beta S^*\mu + \beta I^*\xi$
Assuming equation (3) has pure imaginary roots $\lambda = i\omega(\omega > 0)$, we can obtain

$$-\omega^2 + in_1\omega + n_2 + (\cos(\omega\tau) - i\sin(\omega\tau)) e^{-\mu_1\tau}(in_3\omega + n_4) = 0$$

Separate the real and imaginary parts to obtain:

$$\begin{cases} n_4\cos(\omega\tau) + n_3\omega_1\sin(\omega\tau) = e^{\mu_1\tau}(\omega^2 - n_2) \\ n_3\omega\cos(\omega\tau) - n_4\sin(\omega\tau) = -e^{\mu_1\tau} n_1\omega \end{cases} \tag{10}$$

Add the squares of the two equations, we get $\omega^4 + n_5\omega^2 + n_6 = 0$ where,

$$n_5 = n_1 - 2n_2 - n_3^2 e^{-2\mu_1\tau}, n_6 = n_2^2 - n_4^2 e^{-2\mu_1\tau} \tag{11}$$

Theorem When $\tau = \tau_0$, the Hopf bifurcation occurred near the endemic equilibrium E^*.
Proof By simultaneously solving the two equations presented in (2.3), we can accurately determine the critical value of time delay τ_p:

$$\tau_p = \frac{1}{\omega_1} \arctan \frac{n_2 n_3 \omega_1 - n_3 \omega_1^3}{n_1 n_3 \omega_1^2 + n_2 n_4 - n_4 \omega_1^2} + \frac{p\pi}{\omega_1} \tag{12}$$

where, $p = 0, 1, 2 \ldots$
According to the previous conditions, we can get

$$\omega_0 = \left[\frac{(2n_2 + n_3^2 e^{-2\mu_1\tau} - n_1) + \sqrt{(n_1 - 2n_2 - n_3^2 e^{-2\mu_1\tau})^2 - 4n_2^2 - n_4^2 e^{-2\mu_1\tau}}}{2} \right]^{\frac{1}{2}} \tag{13}$$

Then take the derivative of equation (10) on τ and obtain,

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{n_3\lambda\tau + n_4\tau - n_3 - (2\lambda + n_1) e^{(\mu_1+\lambda)\tau}}{(\lambda+\mu_1)(n_3\lambda + n_4)} \tag{14}$$

Substitute $\lambda = i\omega$, $\tau = \tau_0$, we get

$$\begin{aligned} \mathrm{Re}\left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right] &= \mathrm{Re}\left[\frac{n_3\lambda\tau + n_4\tau - n_3 - (2\lambda + n_1) e^{(\mu_1+\lambda)\tau}}{(\lambda+\mu_1)(n_3\lambda + n_4)} \right] \\ &= \frac{n_3^2\omega^2\mu_1\tau + n_4^2\mu_1\tau - n_3n_4\mu_1 + n_3^2\omega^2}{(\mu_1^2+\omega^2)(n_4^2 + n_3^2\omega^2)} \\ &\quad - \frac{e^{\mu_1\tau}\left[\cos(\omega\tau)(2n_4\omega^2 + 2n_3\mu_1\omega^2 + n_1n_4\mu_1 - n_1n_3\omega^2) \right.}{(\mu_1^2+\omega^2)(n_4^2 + n_3^2\omega^2)} \\ &\quad + \frac{\left. \sin(\omega\tau)(n_1n_4\omega + n_1n_3\mu_1\omega - 2\mu_1n_4\omega + 2n_3\omega^3) \right]}{(\mu_1^2+\omega^2)(n_4^2 + n_3^2\omega^2)} \end{aligned} \tag{15}$$

From the two equations of (15),

$$\begin{cases} \cos(\omega\tau) = \dfrac{e^{\mu_1\tau}(n_4\omega^2 - n_2n_4 - n_1n_3\omega^2)}{n_4^2 + n_3^2\omega^2} \\ \sin(\omega\tau) = \dfrac{e^{\mu_1\tau}(n_3\omega^3 + n_1n_4\omega - n_2n_3\omega)}{n_4^2 + n_3^2\omega^2} \end{cases} \tag{16}$$

Substitute (16) into (17), the transverse condition can be obtained

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right] = \frac{n_3^2\omega^2\mu_1\tau + n_4^2\mu_1\tau - n_3n_4\mu_1 + n_3^2\omega^2}{(\mu_1^2+\omega^2)(n_4^2 + n_3^2\omega^2)} - \frac{e^{2\mu_1\tau}\left[m_1\omega^6 + m_2\omega^4 + m_3\omega^2 + m_4\omega - m_5 \right]}{(\mu_1^2+\omega^2)(n_4^2 + n_3^2\omega^2)^2} \tag{17}$$

where,

$$\begin{aligned} &m_1 = 2n_3^2, m_2 = 2n_4^2 + n_1^2n_3^2 - n_1n_3^2\mu_1 - 2n_2n_3^2, \\ &m_3 = n_1^2n_4^2 - 2n_2n_4^2 - n_1^2n_3n_4\mu_1 - n_1n_2n_3^2\mu_1 - n_1n_4^2\mu_1, \\ &m_4 = n_1^2n_3n_4\mu_1, m_5 = -n_1n_2n_4^2\mu_1 \end{aligned} \tag{18}$$

When $R_0 < 1$, $\beta < (\mu+\xi)e^{\mu_1\tau}$, and $\omega_0 > 0$, we can get

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right] > 0 \tag{19}$$

According to the previous calculation, the system meets the transverse condition for generating Hopf bifurcation, so it can be seen from the Hopf bifurcation theorem of multidimensional time-delay system[6], when $\tau = \tau_0$, the Hopf bifurcation occurred near the endemic equilibrium E^*. The proof is complete.

3. Numerical Simulation

We use ode 45 in MATLAB for numerical simulation. According to the theorem, the endemic equilibrium point is unstable, where the system generates Hopf bifurcation, as shown in the figure 1 and figure 2. From these two figures, we can conclude that the disease is in an oscillatory state, which may be more prone to spiral up, leading to large-scale outbreaks.

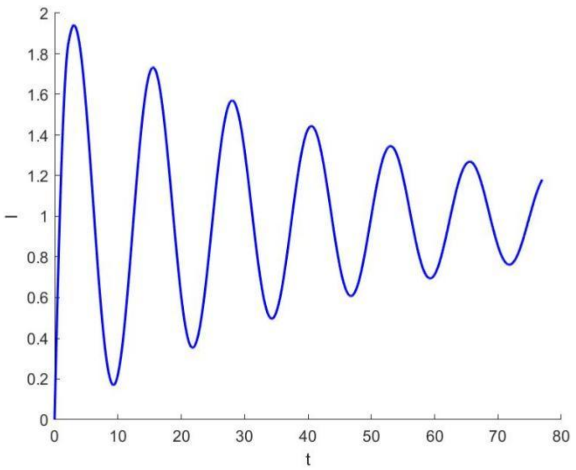

Figure 1. When $\tau > \tau_0$, Time Series Diagram of I.

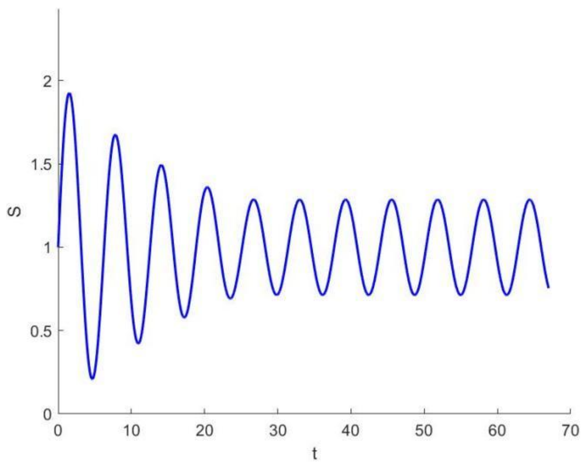

Figure 2. When $\tau > \tau_0$, Time Series Diagram of S.

4. Conclusion

In this paper, the Hopf bifurcation of SIR epidemic model with time delay is deeply studied. Through mathematical analysis and numerical simulation, the complex dynamic behavior and the significance of prevention and control strategies are revealed. The stability of the equilibrium point is analyzed, and the time delay threshold is determined to cause Hopf bifurcation. The dynamic trajectory diagram intuitively shows the behavior of the system which changes at any time. The research enriches the theory of infectious disease dynamics and provides an important reference for the formulation of prevention and control strategies. We firmly believe that with the deepening of research and technological innovation, we will have a more accurate understanding of the spread of infectious diseases, contribute to public health prevention and control, and safeguard human health and safety. This is not only the expansion of scientific knowledge, but also the social responsibility.

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