The Influence of Self-Centralizing Subgroups on the Structure of Finite Groups

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Abstract: In this thesis, we explored the properties of G by using prop-erties of self-centralizing subgroups of G, and we got some new conclusions of SCT-groups. According to the contents, this paper was divided into two parts.In the first part, we mainly gave out concepts of the SCT-groups and introduced their research background and some research results of predecessors.In the second part, we used the self-centralizing subgroups to research the structure of the SCT-groups. If G is an SCT-group then the subgroups and the factor groups of G are also SCT-group. In addition, we have also obtained some other new results, which are detailed in Section 2.

Keywords: Self-Centralizing Subgroups; TI-Subgroups; Nilpotent Groups; SCT-Groups; MSC(2000): 20C15,20D10

1. Introduction

Moving from the part to the whole is one of the most effective ways to understand something. Subgroups, quotient groups, and their arithmetic structures can all be regarded as local aspects of a group. One significant direction and commonly used method in the study of finite group theory is to utilize the properties of subgroups to characterize the structure of groups and explore theirrelated properties. Especially, in the study of finite group structures, the normality of subgroups plays a crucial role in characterizing the properties of groups. N is a normal group of G if and only if $\forall g \in G, N^g = N$. Following the introduction of the concept of normal subgroups, numerous generalizations of the normality concept have emerged. A widely applied conclusion is that a finite group G is nilpotent if and only if all of its maximal subgroups are normal. Researchers in group theory have also obtained many conclusions regarding the impact of normal subgroups on the structure of finite groups. For instance, Baer and Dedekind conducted research on non-abelian groups in which all subgroups are normal subgroups, and such groups are referred to as Hamiltonian groups.Reference derived the structure of finite groups in which all subgroups are either abelian or normal. Thus, it can be seen that normal subgroups have a significant impact on the structure of a group. After the concept of normal subgroups emerged, there were many generalizations of the normal concept. While generalizing the concept of normality, many interesting conclusions were also obtained. For example, Reference presents the structure of finite groups in which subnormal subgroups can replace certain classes of subgroups. TI - subgroups are one such generalization. We refer to a subgroup H as a TI-subgroup of G if for all $g \in G$, either $H^g = H$ or $H^g \cap H = 1$. From the definition, we know that if H is a normal subgroup of G, then necessarily H is a TI-subgroup of G. Many relevant conclusions have also been obtained from studies on TI-subgroups. For instance, Reference characterizes the group structure in which all subgroups are TI - subgroups.Hochheim and Timmesfeld studied the group structure with only one abelian $TI-2$-subgroup in Reference. In recent years, Guo.et.al provided a complete classification of groups in which all abelian subgroups are TI-subgroups in Reference. Semi-normal subgroups represent another generalization of normal subgroups. As is well-known, the center of a group and its centralizer play significant roles in the study of finite groups. In recent years, some authors have studied the group structure where certain specific subgroups are self-centralizing. For example, Delizia.et.al classified finite groups in which all non-cyclic subgroups are self-centralizing in Reference. Reference conducted in-depth research on finite groups in which all non-abelian subgroups are self-centralizing, yielding many new results.Many group theorists have focused their research on the structure of finite 2-groups and finite 3-groups in which all non-abelian subgroups are self-centralizing.

Naturally, we began to think in reverse: What is the structure of a finite group when all self-centralizing subgroups are TI -subgroups? And what is the structure when all self-centralizing subgroups are semi-normal subgroups?In Reference, Mahmoud introduced a new definition: a finite group in which all self-centralizing subgroups are normal is called an SCN-group.Of course, SCN-groups do exist, and abelian groups and Hamiltonian groups are two obvious examples of them.

Inspired by literature, we can consider the structure of finite groups in which all self-centralizing subgroups are TI -subgroups. We shall refer to such groups as SCT -groups.Our main findings reveal that finite SCT-groups are either nilpotent groups or F-groups, among other possibilities. It is evident from the definition of SCN -groups that SCN -groups must be SCT -groups, confirming the existence of the class of groups considered in this paper.

Throughout this paper, all groups are assumed to be finite. G/N is denoted the quotient group ofGmodulo N.G'represents the derived subgroup of G. All unspecified symbols and notations follow standard conventions and can be referenced from Xu Mingyao's work.

The first part of this chapter primarily introduces the research background and prerequisite knowledge related to groups, while the second part primarily presents relevant definitions and theorems of groups.

In this chapter, we primarily investigate the structure of finite groups in which all self-centralizing subgroups are either TI -subgroups or seminormal subgroups. We term a finite group whose all self-centralizing subgroups are TI-subgroups as an SCT-group. Utilizing specific properties of certain special subgroups, we delve into the structure of finite groups and primarily obtain several novel results regarding SCT -groups, including the inheritance of subgroup and quotient group properties, as well as the characterization of SCT -groups as nilpotent groups or F-groups. This chapter is divided into two main sections. The first section provides fundamental definitions of SCT -groups and introduces the lemmas used in the proofs that follow. The second section presents several novel results pertaining to the properties and characterizations of SCT-groups.

2. SCT-Groups

2.1 Some Basic Definitions and Lemmas

In this section, there are some relevant definitions and useful lemmas of SCT-groups.

Definition 2.1.1 Let G be a finite group. H is a Self-centered subgroup of G if H satisfies $C_G(H) \subseteq H$.

Definition 2.1.2 Let G be a finite group. G is defined as a $SCN-$ group if its Self-centered subgroups are normal.

Definition 2.1.3 Let G be a finite group. H is defined as a TI-subgroup of G if for every $g \in G$ satisfies $H^g = H$ or $H^g \cap H = 1$.

Definition 2.1.4 Let G be a finite group. G is defined as a $CN-$ group if its non-identity element Self-centered subgroups are nilpotent.

Definition 2.1.5 Let G be a finite group. G is defined as a $SCT-$ group if its Self-centered subgroups are TI-subgroups.

Lemma 2.1.1 Let G be a finite group and $H \leq G$. If K is a self-centered subgroup of G, then $\widehat{K} =< K, C_G(K) >$ is a self-centered subgroup of G, and the following statements hold.

(1)$K \cap H = K$;

(2)$N_G(\widehat{K}) \cap H \leq N_H(K)$.

Lemma 2.1.2 (Rocke) Let G be a finite p-group and for every $g \in G$, $C_G(g) = \{a \in G | ag = ga\}$ is normal, then $cl(G) \leq 2$ or G is a finite 3-group with 3 conjugated classes.

Lemma 2.1.3 Let G be a finite group and $H \leq G$. Then $N_G(H)$ is a self-centralizing subgroup in G.

Lemma 2.1.4 Let G be a finite group. All Self-centralizing subgroups of G are normal subgroups if and only if G is nilpotent with at most two nilpotent classes.

2.2 Some New Results on SCT-Groups

In studying SCT -groups, we firstly consider whether they possess properties such as subgroup inheritance and quotient group inheritance, and then further investigate the specific properties of SCT-groups. The following new results are obtained.

Theorem 2.2.1. Let G be a finite group and N is a normal subgroup of G. If G is a SCT-group, Then H/G is also a SCT-group.

Proof. Let $K \leq H$ be a self-centralizing subgroup in H. $\widehat{K} =< K, C_G(K) >$ is a self-centralizing subgroup in G by Lemma 2.2.1.Thus, $\widehat{K}$ is a TI-subgroup of G.Namely, for every $g \in G$, $\widehat{K}_g \cap \widehat{K} = \widehat{K}$ or $\widehat{K}_g \cap \widehat{K} = 1$. Now, it will be proved that K is a TI-subgroup of H. If there exists a $h \in H$ such that $\widehat{K}_g \cap \widehat{K} \neq \widehat{K}$ and $\widehat{K}_g \cap \widehat{K} \neq 1$, then we obtain $h \notin H$ from $N_G(\widehat{K}) \cap H = N_H(K)$ by Lemma 2.1.1(2). Therefore, $\widehat{K}_g \cap \widehat{K} \neq \widehat{K}$ and $\widehat{K}_g \cap \widehat{K} \neq 1$ leads to a contradiction.

Theorem 2.2.2 Let G be a finite group and N is a normal subgroup of G. If G is a SCT-group, Then H/G is also a SCT-group.

Proof. Let H/N be a self-centralizing subgroup in G/N. Then $C_G(H) \leq H$ from

$$C_G(H)/N \leq C_{G/N}(H/N) = H/N$$

Namely, H is a self-centralizing subgroup of G. G is a SCT-group thus for every $g \in G$, $H^g \cap H = H$ or $H^g \cap H = 1$. Now, it will be proved that H/N is a TI-subgroup of G/N. If there exists a $gN \in G/N$ such that

$$(H/N)^{gN} \cap H/N \neq H/N \text{and} (H/N)^{gN} \cap H/N \neq N,$$

Then,

$$(H/N)^{gN} \cap H/N = H^g/N \cap H/N \neq H/N \text{and} N,$$

Namely, $H^g \cap H \neq H$ and $H^g \cap H \neq 1$. This is a contradiction with that H is a TI-group of G. Proven.

Theorem 2.2.3 Let G be a finite SCT-group. $N \trianglelefteq G$ and H is a self-centralizing subgroup in G. Then G/N is an Abelian group or Hamilton group. Especially, $|G|$ is a odd number, then G/N is an Abelian group.

Proof. Let $N \leq H \leq G$. If N is self-centralizing,then H is a self-centralizing group of G by

$$C_G(H) \leq C_G(N) \leq N \leq H.$$

Because G is a SCT-group, therefore H is a TI-subgroup of G. For every $g \in G$, $N \leq H^g \cap H$ is established by $N = N^g \leq H^g$. According to the definition of a TI-subgroup, $H^g \cap H = H$ is established, namely,$H \trianglelefteq G$. Thus $H/N \trianglelefteq G/N$.I.e any subgroup of G/N is normal. Proven.

By the above G/N has been proved an Abelian group or Hamilton group. If G is a Hamilton group, then $G/N = Q_8 \times A \times B$, thereinto Q_8 is a Four-element group, A is an abelian group of odd order, the fact that B is an elementary abelian 2-group contradicts the fact that the order of G is odd.

Theorem 2.2.4 If G be an SCT-group with odd order, then $G' \leq Z(G)$.

Proof. Let $a \in G$. Based on $C_G(C_G(a)) \leq C_G(a)$ we can conclude that $C_G(a)$ is a self-centralizing group of G.$C_G(a)$ is a TI-subgroup according to G is an SCT-group. Furthermore, because $Z(G) \leq C_G(a)$, for every $g \in G$,$Z(G) \leq C_G(g)^g \cap C_G(a)$, namely, $C_G(a) \trianglelefteq G$. According to the theorem 2.2.3, $G/C_G(a)$ is commutative, thus $G' \leq C_G(a)$. Due to the arbitrariness of a, it follows that $G' \leq C_G(a)$ holds true.

The above results discuss the case where, when G is an SCT-group of odd order, it follows that $G' \leq Z(G)$. To ensure the completeness of the discussion, we need to determine whether SCT-groups of even order possess similar properties.Next, we will further determine the properties of SCT-groups of even order by discussing whether finite 2-groups that are SCT-groups possess similar properties.

Theorem 2.2.5 If G is finite 2-group and an SCT-group, then $G' \leq Z(G)$.

Proof. According to the Theorem 2.2.4 $C_G(a)$ is self-centralizing in G for every $a \in G$ .And $C_G(a) \trianglelefteq G$ because $Z(G) \leq G$ and G is an SCT-group. $cl(G) \leq 2$ or G is finite 3-group by Lemma 2.1.2 Tus $cl(G) \leq 2$ due to that G is a finite 2-group. Namely, $G' \leq Z(G)$.

Based on the above, we can easily draw the following inference.

Corollary 2.2.1 If G is an SCT-group, then $G' \leq Z(G)$.

Corollary 2.2.2 If G is an SCT-group, then $Inn(G)$ is an Abelian group.

Proof. Based on Inference 2.2.1, $G/Z(G)$ is an Abelian group. $Inn(G)$ is commutative because $G/Z(G) = Inn(G)$.

Theorem 2.2.6 G is a finite SCT-group, then one of the following statements holds true.

(1)G is a nilpotent group.

(2)G = NH is an F-group, N is the F-kernel, H is the F-complement. And N is the unique minimal normal subgroup of G. H is nilpotent. Particularly, G is a solvable CN-group.

Proof. Assuming that G is a non-nilpotent group, then G has at least one non-normal maximal subgroup, denoted as H .Obviously, we have $(C_G(H) \leq N_G(H))$, thus H is a self-centralizing subgroup. Since G is an SCT-group, it follows that H is a TI-subgroup. According to [11, V. Hauptsatz 7.6], G is an F-group with H as a complement. Let N be the F-kernel of G and V be a minimal normal subgroup of G. Then $V \subseteq N$ is nilpotent (see [11, V, Satz 8.7]). Therefore, V is an elementary abelian group whose order is p^n, where p is a prime number. Further, $H \cong G/V$ holds on. Let K/V be any maximal subgroup of G/V. Then K is a maximal subgroup of G. If K is not normal in G, then K is a self-centralizing subgroup of G, and hence a TI-subgroup. But we know that K contains the normal subgroup V of

G. By the definition of a TI-subgroup, this would imply that K is a normal subgroup of G, which contradicts our assumption. That is, K must be normal in G. Consequently, K/V is a normal subgroup of G/V. Then G/V is a nilpotent group, and hence H is nilpotent. By the definition of a CN-group, it immediately follows that G is a solvable CN-group.

Theorem 2.2.7 Let G is a finite nilpotent group. Then, all the self-centralizing subgroups of G are TI-subgroups if and only if $cl(G) \leq 2$.

Proof. Let $G = P_1 \times P_2 \times \cdots \times P_s$, thereinto, $P_i \in Syl_{p_i}(G)$ and $p_i \in \pi(G)$. Firstly, let's assume that all centralizer-free subgroups in G are TI-subgroups. According to Theorem 2.1.1, all self-centralizing subgroups in P_i are also TI-subgroups. Based on Corollary 2.1.1, we have $P_i' \leq Z(P_i)$. From this, The following equation holds.

$$G' = P_1' \times P_2' \times \cdots \times P_s' \leq Z(P_1) \times Z(P_2) \times \cdots \times Z(P_s) \leq Z(G)$$

Conversely, assuming that $cl(G \leq 2$ and H is self-centralizer of G then

$$G' \leq Z(G) \leq C_G(H) \leq H.$$

Thus H is a normal subgroup of G.According to the definition, H is a TI-subgroup in G.

Corollary 2.2.3 All self-centralizing subgroups of G are normal if and only if G is nilpotent and $cl(G) \leq 2$ holds on.

Proof. Assuming that all self-centralizing subgroups of G are normal but G is non-nilpotent. Then G has at least one non-normal maximal subgroup, denoted as H.Then $N_G(H) = H$ and H is self-centralizing in G. Thus H is normal in G. That is G is nilpotent. The proof of the other part of this corollary is the same as that of Theorem 2.1.7, and will not be repeated here.

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Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

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