Analysis of the Construction Thought of Generating Functions

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Abstract: In this paper, we conduct a systematic analysis of the construction thought for generating functions. We specifically examine the construction processes of specific algebraic forms, demonstrating their practical advantages through combinatorial counting problems. We extend the structural analysis of generating functions through the integration of Euler's proof of the infinitude of primes. Furthermore, we interpret the Bertrand-Chebyshev theorem from the perspective of generating functions, highlighting how the absence of elementary contributions constrains the representational power of generating functions. Through an in-depth analysis of challenging sequence problems, we demonstrate the potential of generating functions in addressing complex mathematical problems. Finally, the future development of generating function construction thought is prospected, underscoring its indispensable role in modern mathematics and interdisciplinary fields.

Keywords: Construction Thought; Generating Functions; Representational Power; Sequence

1. Introduction

As a core analytical tool in discrete mathematics, generating functions have profoundly influenced the developmental trajectories of combinatorics and number theory since their conceptual inception. The origins of this concept can be traced to Euler's systematic application of power series, where he encoded numerical sequences into continuous functions to solve recurrence relations and partition problems, thereby establishing the methodological foundation of generating functions. Subsequent scholars, including Laplace, Ramanujan, and Pólya, further expanded their applications, solidifying their irreplaceable role in enumerative combinatorics, probability theory, and algorithmic analysis.

Bach E's investigation of generating functions and polynomial division serves as an excellent paradigm for our research [1]. Recent studies by Gu et al. [2], Xu et al. [3] and Zhu et al. [4] have investigated the formal expressions of generating functions, while Han et al. investigated the recursive solution of the double forcing polynomial for ladder graphs [5]. Qi et al. explored their recursive formulations in subdivision schemes [6]. However, current research still lacks a systematic analysis of the under-lying construction thought of generating functions. Classical literature predominantly focuses on surface-level applications to counting problems (e.g., integer partitions and Catalan numbers), leaving the potential of generating functions in complex discrete systems underexplored. For instance, Euler's proof of the infinitude of primes, though not explicitly framed in generating function terminology, implicitly embodies the conceptual framework of linking discrete sequences through analytic continuation. The recent surge in research on nonlinear recursions and combinatorial optimization problems has been particularly notable. Numerous scholars have conducted research on generation [7] and recursion [8,9], as well as on polynomials. Zhang investigates the sequence model from the perspective of generating functions [10]. Although generating functions represent a powerful tool for solving highly complex sequence problems, the application of such techniques in sequence-related issues remains challenging without a solid grasp of construction thought [11-13]. Moreover, our analysis reveals that the Bertrand-Chebyshev theorem inherently demonstrates how gaps in prime distribution—manifested as missing elemental contributions—constrain the representational capacity of generating functions, thereby offering a critical entry point for methodological innovation.

To address these critical gaps, this paper aims to systematically elucidate the core thought underlying the construction of generating functions and illuminate their methodological value in discrete problem-solving [14] through representative case studies.

2. Process and Analysis

2.1 Some Examples

Generating functions hold significant applications in solving discrete problems. This can be illustrated through several specific examples.

For partition problems, generating functions can be employed to calculate different ways of partitioning integers. The generating function for integer partitions is ex-pressed as:

$$P(x)=\prod_{k=1}^{\infty}\frac{1}{1-x^k} \tag{1}$$

In this context, the coefficient represents the partition number of different integers k . For example, the coefficient of the term x^5 indicates the number of ways to partition 5 into a sum of distinct integers. Using this generating function, partition numbers can be computed efficiently without enumerating all possible partitions.

Generating functions are also widely applied in calculating Catalan numbers, playing a crucial role particularly in counting problems like binary tree structures and parenthesis matching. For instance, with the generating function:

$$C(x)=\frac{1-\sqrt{1-4x}}{2x} \tag{2}$$

Various combinatorial numbers can be calculated clearly.

Returning to one of the classic problems in combinatorics—the Fibonacci sequence [15]—its recurrence relation is:

$$\begin{cases} F(n)=F(n-1)+F(n-2) \\ F(0)=0 \\ F(1)=1 \end{cases} \tag{3}$$

Using the generating function, the recurrence relation can be transformed into an algebraic equation:

$$G(x)=\frac{x}{1-x-x^2} \tag{4}$$

Through the generating function, one can easily derive the explicit formula without recursive computation term by term.

Generating functions also have extensive applications in probability theory, aiding systematic calculation and analysis of probability distributions. For a discrete random variable X , its probability generating function is defined as:

$$G_X(x)=E[x^X]=\sum_{k=0}^{\infty}P(X=k)x^k \tag{5}$$

By differentiating the generating function, statistical measures such as the expected value and variance of the random variable can be directly obtained. Take a random variable $X\sim Bin(n,p)$ that follows a binomial distribution as an example; its probability generating function is:

$$G_X(x)=(px+1-p)^n \tag{6}$$

This generating function provides a convenient method to calculate various properties of the binomial distribution. Without summing term by term, it significantly improves problem-solving efficiency.

2.2 Construction of Specific Forms

As a classic early application of the generating function idea, the construction of the generating function for the Fibonacci sequence problem demonstrates the ingenuity of its application. To deeply explore the construction of specific forms [16], an original self-designed problem is as follows:

Given the sequence $\{a_n\}$ satisfies $a_n=a_{n-1}+2a_{n-2}$, with $a_0=1$ and $a_1=3$, prove that for all $n\geq 0$, $a_n\leq 3^n$ holds.

This problem can be addressed using mathematical induction. However, here we illustrate the construction of a generating function to establish a connection between the discrete and the continuous domains. Define the generating function:

$$\begin{aligned} A(x)&=a_0+a_1x+a_2x^2+a_3x^3+\cdots \\ &=\sum_{n=0}^{\infty}a_nx^n \end{aligned} \tag{7}$$

Using the recurrence relation $a_n=a_{n-1}+2a_{n-2}$, we transform (7) as follows:

$$\begin{aligned} A(x)&=a_0+a_1x+\sum_{n=2}^{\infty}a_nx^n=a_0+a_1x+\sum_{n=2}^{\infty}(a_{n-1}+2a_{n-2})x^n=a_0+a_1x+\sum_{n=2}^{\infty}a_{n-1}x^n+2\sum_{n=2}^{\infty}a_{n-2}x^n \\ &=a_0+a_1x+x\left(A(x)-a_0\right)+2x^2A(x)=1+3x+xA(x)-x+2x^2A(x) \end{aligned} \tag{8}$$

After rearranging equation (8), we derive:

$$A(x)=\frac{1+2x}{1-x-2x^2} \quad (9)$$

Next, define another generating function:

$$B(x)=\sum_{n=0}^{\infty}3^n x^n=\frac{1}{1-3x} \quad (10)$$

It is necessary to prove that $A(x)$ in equation (9) is smaller than $B(x)$ in equation (10) when $|x|<\frac{1}{3}$. Eventually, simplifying the inequality leads to $x^2 \geq 0$, which obviously holds. Thus, the proposition is proved.

2.3 Applying the Construction Thought to Practical Problems

In this section, we employ generating functions as a tool to address practical problems. Consider the following problem designed for illustration:

There exist five distinct types of balls, each available in an infinite quantity and differing in color. The task is to select 10 balls under the constraint that at least one red ball must be included. Determine the number of possible ways to perform this selection.

According to the problem, since at least one red ball must be selected, the possible numbers of red balls to select are 1, 2, 3,…, and its generating function is:

$$G_1(x)=x+x^2+x^3+\cdots=\frac{x}{1-x} \quad (11)$$

For the balls of the remaining four colors, each color allows selecting 0 or more balls. Thus, the generating function for each color is:

$$G_2(x)=1+x+x^2+x^3+\cdots=\frac{1}{1-x} \quad (12)$$

Here, x^k represents selecting k balls. As the selections of different colors are independent, the overall generating function $G(x)$ is the product of the generating functions of each color:

$$\begin{aligned}G(x)&=G_1(x)\left(G_2(x)\right)^4\\&=\frac{x}{1-x}\cdot\left(\frac{1}{1-x}\right)^4=\frac{x}{(1-x)^5}\end{aligned} \quad (13)$$

To find the total number of ways to select 10 balls, this corresponds to the coefficient of x^{10} in the generating function $G(x)$, that is:

$$[x^{10}]\frac{x}{(1-x)^5}=[x^9]\frac{1}{(1-x)^5} \quad (14)$$

To find the coefficient of x^9, the extended form of the Binomial Theorem is used:

$$\frac{1}{(1-x)^k}=\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n \quad (15)$$

Substitute $k=5$ and $n=9$:

$$[x^9]\frac{1}{(1-x)^5}=\binom{9+5-1}{5-1}=715 \quad (16)$$

Therefore, under the condition of including at least one red ball, there are 715 different ways to select 10 balls from 5 types of balls with infinite quantities in different colors.

2.4 The Deepening of Thought

Euler’s proof of the theorem on the infinitude of prime numbers is renowned for its classical nature and innovative features. Although it does not strictly belong to the traditional generating function category, it profoundly embodies the idea of generating functions. Euler’s method transcends the counting level. Through the decomposition properties of generating functions, it reveals structural problems in number theory, marking the deepening and expansion of the generating function idea from counting analysis to structural research.

In Euler’s proof, the generating function appears in the form of the Riemann function:

$$\zeta=\sum_{n=1}^{\infty}\frac{1}{n^s} \quad (17)$$

Where $s>1$ ensures the convergence of the series. This function captures the structure of all natural numbers, where each term n^{-s} corresponds to the contribution of the natural number n. By uniquely decomposing each n into prime power products:

$$n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k} \quad (18)$$

Euler transformed the generating function of natural numbers into a generating function related to prime numbers, thus obtaining the Euler product form of the Riemann function:

$$\begin{aligned}\zeta(s)&=\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\\&=\prod_p\frac{1}{1-p^{-s}}\end{aligned} \quad (19)$$

Where p represents prime numbers. This decomposition directly links the generating function of natural numbers with the structure of prime numbers, indicating that the structure of natural numbers is completely composed of prime numbers.

Further considering the limit of $\zeta(s)$ as $s \to 1^+$, at this time, $\zeta(s)$ degrades into the harmonic series:

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} \tag{20}$$

The harmonic series is divergent, indicating that $\zeta(s)$ tends to infinity at $s = 1$. However, if the number of prime numbers were finite, assuming the prime set is a finite set $\{p_1, p_2, p_3, \ldots, p_k\}$, then the Euler product form of $\zeta(s)$ would be:

$$\zeta(s) = \prod_{i=1}^{k} \frac{1}{1 - p_i^{-s}} \tag{21}$$

The product of a finite number of terms results in a finite value. Even as $s \to 1^+$, this product would also converge to a finite value. This contradicts the divergent property of the harmonic series $\zeta(1)$. Therefore, the set of prime numbers cannot be finite, and thus it is concluded that prime numbers are infinite.

Euler's proof encodes the structures of natural numbers and prime numbers into the $\zeta(s)$ function and its Euler product form. By analyzing the divergence of the generating function and the contradiction in the decomposition form, it intuitively proves the infinitude of prime numbers, demonstrating the powerful role of generating functions in discrete mathematics. Next, we will explore its underlying mechanism and core ideas to provide a clearer theoretical perspective for revealing the essence.

2.5 Core of the Construction Thought

Ling and Xu [17] stated, "The step-by-step multiplication counting principle essentially illustrates the necessary procedures for completing a task, progressing step by step with interlocked links. These two counting principles, combined with counting formulas, form the core process by which generating functions handle counting problems."

The core idea of using generating functions to address counting problems can be further generalized to the construction thought in generating functions. Its essence lies in systematically linking discrete elements in an ordered sequence through generating functions. This process can be metaphorically likened to threading multiple rings onto a string: to retrieve a specific ring, the preceding ones must be removed sequentially, with each step laying the foundation for subsequent actions. The absence of any step disrupts the entire sequence and impacts later stages. In this analogy, forcibly extracting a ring from the back would deform the structure, stripping the rings of their original form and properties. For generating functions, this corresponds to the loss of contributions from certain elements, thereby weakening the function's representational capacity and failing to preserve the intrinsic characteristics of the combinatorial structure.

To intuitively demonstrate this core idea, consider a specific example. Using the construction thought of generating functions, we explore the Bertrand-Chebyshev theorem's core implication (discussed here, without rigorous proof). The theorem states: for each integer $n > 1$, there is at least one prime in $(n, 2n)$.

Define a generating function $F(x)$ to encode prime number information within $2n$:

$$F(x) = \prod_{p \le 2n} \frac{1}{1 - x^p} = \sum_{k=0}^{\infty} a_k x^k \tag{22}$$

Where p is prime number, and a_k denotes the number of ways to express integer k as a sum of primes. We decompose the expression in equation (22) into two parts:

First, we explore the actual expression form of the generating function. Let q be the largest prime in $(0, n)$. A detailed expansion of $F_1(x)$ in equation (23) is as follows:

$$\begin{aligned} F(x) &= \prod_{p \le 2n} \frac{1}{1 - x^p} \\ &= \prod_{p \le n} \frac{1}{1 - x^p} \cdot \prod_{n < p \le 2n} \frac{1}{1 - x^p} \\ &= F_1(x) \cdot F_2(x) \end{aligned} \tag{23}$$

$$\begin{aligned} F_1(x) &= \prod_{p \le n} \frac{1}{1 - x^p} \\ &= \left(\frac{1}{1 - x^2}\right)\left(\frac{1}{1 - x^3}\right)\cdots\left(\frac{1}{1 - x^q}\right) = (1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots(1 + x^q + x^{2q} + \cdots) \end{aligned} \tag{24}$$

From equation (24), it is evident that each prime number recorded in the generating function is formed by the permutation and combination of the preceding sequences. Each prime number serves as a base point, gradually constructing new combinations.

Next, explore the generating function's cumulative contribution. By absolute convergence, let $k = mp$:

$$\log F_1(x) = \log\left(\prod_{p\le n}\frac{1}{1-x^p}\right) = \sum_{p\le n}\log\left(\frac{1}{1-x^p}\right)$$
$$= \sum_{p\le n}\sum_{m=1}^{\infty}\frac{x^{mp}}{m} = \sum_{m=1}^{\infty}\sum_{p\le n}\frac{x^{mp}}{m} = \sum_{k=1}^{\infty}\left(\sum_{\substack{p\le n\\ p|k}}\frac{p}{k}\right)x^k = \sum_{k=1}^{\infty}c_k x^k \tag{25}$$

Apply the exponential operation to (25) to restore the form of the generating function:

$$F_1(x) = e^{\sum_{k=1}^{\infty}c_k x^k} = \left(1 + c_1 x + \frac{(c_1 x)^2}{2!} + \frac{(c_1 x)^3}{3!} + \cdots\right)\left(1 + c_2 x + \frac{(c_2 x)^2}{2!} + \frac{(c_2 x)^3}{3!} + \cdots\right)\cdots \tag{26}$$

Observing (26), if there are no prime numbers in the interval $(n, 2n)$, its cumulative effect will gradually weaken, leading to a significant decline in generating capability—i.e., the generation of larger integers becomes increasingly sparse. Thus, we can intuitively understand how the generating function transforms discrete problems into continuous ones. The core of the generating function construction thought lies in the profound logic behind it and the unique charm of its mathematical structure.

The concepts of constructive thought and representational ability may be somewhat abstract. To help readers understand, we present a simple example to vividly demonstrate this process:

Based on the understanding of the core of generating functions and constructive thinking, an extension is made. By mimicking the above thinking, another method is adopted to interpret the Bertrand-Chebyshev theorem. For the research needs of this paper, we assume here that the Goldbach Conjecture holds—any even number greater than 2 can be expressed as the sum of two prime numbers. A shift in thinking is required here, imitating the core of generating functions and constructive thought: "Each step lays the foundation for subsequent steps, and the absence of any step affects the follow-up processes [18]." The premise for seeking the next even number is that two prime numbers have already been found to represent the currently selected even number.

Starting with the even number $2n$, it is known that $2n = n + n$. If n is a prime number, consider $k = 2n + 2$. Obviously, the combination of these two prime numbers will not be $2n$ and 2. The two prime numbers can then be written as $(2 + a)$ and $(2n - a)$, where $0 < a < 2n - 1$ and a is an integer. Notably, at least one of them lies in the interval $(n, 2n)$. If n is not a prime number, continue to seek two prime numbers to represent $2n$. $2n$ can be expressed as the sum of two prime numbers $(n - b)$ and $(n + b)$, where $0 < b < n - 1$ and b is an integer. Again, at least one lies in the interval $(n, 2n)$. Therefore, there exists at least one prime number within the interval $(n, 2n)$.

2.6 Highly Complex Construction

The core role of generating functions is primarily manifested in the solution of sequence problems, particularly serving as an essential problem-solving tool for highly complex sequences—especially non-arithmetic or non-geometric sequences [19]. Inspired by Zhu, we posit that the construction thought of generating functions in sequence problems lies in perceiving the commonalities and differences between sequences and functions, thereby appreciating the holistic nature of mathematics [20]. The construction thought is not limited to the examples mentioned earlier; in those examples, the construction of generating functions is not complex. The real difficulty lies in understanding the structure and mechanism of generating functions, as well as the specific meanings of their coefficients and degrees in the problems. The key is how to think of using generating functions to solve problems.

For many scholars, it is not easy to think of using generating functions to solve sequence problems. This ability depends on rich problem-solving experience or an understanding of the mathematical culture and history of generating functions [21], especially in classic problems such as the Fibonacci sequence. In high-difficulty sequence problems, the use of generating functions can be divided into two categories: one is that simply thinking of generating functions allows for a smooth construction, testing divergent thinking and flexibility; the other requires the combination of divergent thinking and convergent thinking. Even if generating functions are thought of, it is difficult to construct them accurately, demanding that problem solvers possess mathematical intuition and strong construction abilities.

Inspired by Li [22-25], the self-designed original question is as follows:

Given the sequence $\{a_n\}$ satisfying $a_{n+1}+a_n-2a_{n-1}=\frac{(3n+2)(n+1)}{2}$, with $a_0=0$ and $a_1=1$, find the general term formula of $\{a_n\}$.

Define the generating function $f(x)=\sum_{n=0}^{\infty}a_nx^n$. According to the recurrence relation $a_{n+1}+a_n-2a_{n-1}=\frac{(3n+2)(n+1)}{2}$, multiply both sides by x^n and sum over n from 1 to ∞ to construct the generating function, obtaining:

$$\sum_{n=1}^{\infty}a_{n+1}x^n+\sum_{n=1}^{\infty}a_nx^n-2\sum_{n=1}^{\infty}a_{n-1}x^n=\sum_{n=1}^{\infty}\frac{(3n+2)(n+1)}{2}x^n \tag{27}$$

Handle each term on the left side of equation (27) separately. For the first term:

$$\begin{aligned}\sum_{n=1}^{\infty}a_{n+1}x^n&=\frac{1}{x}\sum_{n=1}^{\infty}a_{n+1}x^{n+1}\\&=\frac{f(x)-a_0-a_1x}{x}\end{aligned} \tag{28}$$

Substitute $a_0=0$ and $a_1=1$ into (28), we get:

$$\sum_{n=1}^{\infty}a_{n+1}x^n=\frac{f(x)}{x}-1 \tag{29}$$

Processing the second term on the left side of the equal sign in (27):

$$\sum_{n=1}^{\infty}a_nx^n=f(x)-a_0=f(x) \tag{30}$$

The third term:

$$\sum_{n=1}^{\infty}a_{n-1}x^n=x\sum_{n=0}^{\infty}a_nx^n=xf(x) \tag{31}$$

Combining and sorting out (29), (30) and (31), we can obtain:

$$\sum_{n=1}^{\infty}a_{n+1}x^n+\sum_{n=1}^{\infty}a_nx^n-2\sum_{n=1}^{\infty}a_{n-1}x^n=\left(\frac{1}{x}+1-2x\right)f(x)-1 \tag{32}$$

Next, process the right side of the equal sign in (27):

$$\sum_{n=1}^{\infty}\frac{(3n+2)(n+1)}{2}x^n=\sum_{n=1}^{\infty}\frac{3n^2+5n+2}{2}x^n=\frac{3}{2}\sum_{n=1}^{\infty}n^2x^n+\frac{5}{2}\sum_{n=1}^{\infty}nx^n+\sum_{n=1}^{\infty}x^n \tag{33}$$

Using the known generating function formulas:

$$\begin{cases}\sum_{n=1}^{\infty}x^n=\frac{x}{1-x}\\ \sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2}\\ \sum_{n=1}^{\infty}n^2x^n=\frac{x(1+x)}{(1-x)^3}\end{cases} \tag{34}$$

Substituting (33) and conducting the simplification, we obtain:

$$\sum_{n=1}^{\infty}\frac{(3n+2)(n+1)}{2}x^n=\frac{x^3-3x^2+5x}{(1-x)^3} \tag{35}$$

Using the results from (32) and (35), we establish an equation for $f(x)$ and simplify it step by step:

$$\begin{aligned}&\left(\frac{1}{x}+1-2x\right)f(x)-1=\frac{x^3-3x^2+5x}{(1-x)^3}\\&\left(\frac{1}{x}+1-2x\right)f(x)=\frac{x^3-3x^2+5x+(1-x)^3}{(1-x)^3}\\&\left(\frac{1}{x}+1-2x\right)f(x)=\frac{2x+1}{(1-x)^3}\end{aligned} \tag{36}$$

Finally, solving yields:

$$f(x)=\frac{x(2x+1)}{(2x+1)(1-x)^4}=\frac{x}{(1-x)^4} \tag{37}$$

This represents the final closed-form of the generating function.

Next, expand the generating function to find the general term. It is known that $\frac{1}{(1-x)^4}=\sum_{n=0}^{\infty}\binom{n+3}{3}x^n$. Thus:

$$f(x)=x\sum_{n=0}^{\infty}\binom{n+3}{3}x^n=\sum_{n=0}^{\infty}\binom{n+3}{3}x^{n+1} \tag{38}$$

This indicates that for $n\geq 1$, the general term formula of the sequence $\{a_n\}$ is:

$$a_n=\binom{n+2}{3}=\frac{n(n+1)(n+2)}{6} \tag{39}$$

It is found that when $n=0$, $a_0=0$ also holds. Therefore, (39) is the general term formula of the sequence $\{a_n\}$.

Some scholars may attempt to solve this

problem through general methods such as the method of superposition. Such methods usually introduce the difference sequence $b_n = a_{n+1} - a_n$ based on the recurrence relation, gradually solve the expression of b_n, and finally restore the general term formula of a_n through superposition. However, the recurrence relation of this problem is not in the simple form of $b_n - b_{n-1}$ and cannot be directly superimposed. Solving the general term formula of b_n involves the particular solution of the non-homogeneous equation [26], constructing complex mathematical techniques, which requires extremely high comprehension and problem-solving skills.

3. Conclusions and Discussion

For highly difficult sequence problems, there are various construction methods for generating functions. Here, we discuss another case:

Given $b_0 = 3$ and $b_1 = 9$, satisfying $b_{n+2} = 4b_{n+1} - 3b_n - 4n - 6$, find the general term formula of $\{b_n\}$.

The construction method for this problem is as follows:

$$b_{n+2} = 4b_{n+1} - 3b_n - 4n - 6 \;\Updownarrow\; b_n = 4b_{n-1} - 3b_{n-2} - 4n + 2, n \geq 2 \quad (40)$$

$$f(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=2}^{\infty} b_n x^n + b_1 x + b_0 = \sum_{n=2}^{\infty} (4b_{n-1} - 3b_{n-2} - 4n + 2)x^n + 9x + 3 \quad (41)$$

It is evident that this directly transforms $f(x)$, eventually written as Expression (41). Through step-by-step simplification, substituting (41) with an expression in terms of x and $f(x)$, we finally obtain:

$$(1 - 4x + 3x^2) \cdot f(x) = \frac{2 - 6x + (1-x)^3}{(1-x)^2} \quad (42)$$

The difference in the construction thought of this sequence problem from previous sequence problems lies in that, for previous sequence problems, the construction was directly applied to the relational expression:

$$\begin{cases} \sum_{n=1}^{\infty} a_{n+1}x^n + \sum_{n=1}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} \frac{(3n+2)(n+1)}{2} x^n \\ f(x) = \sum_{n=0}^{\infty} a_n x^n \end{cases} \Rightarrow f(x) = ? \quad (43)$$

The complexity of the recurrence relation (whether it contains non-homogeneous terms, high-order terms, or non-linear terms) determines the construction difficulty of the generating function. The core difference between these two cases is that the degrees of the polynomials vary, leading to different complexities in splitting non-homogeneous terms. Quadratic terms require combining high-order generating functions, while linear terms only need low-order expansions. Generating functions offer a unified framework that elucidates the underlying connections among problems, thereby integrating and unifying many ostensibly unrelated mathematical issues. For highly challenging sequence problems, we encapsulate the advantages of generating functions in one statement: through algebraic transformations, discrete problems are reformulated as continuous function problems, circumventing direct computations of recursive superposition. This approach is particularly suited for efficiently solving complex recurrence relations. Regardless of how the recurrence structure evolves, generating functions translate the complexity of recurrence relations into equation-solving problems via algebraization and series operations. The ultimate objective is to simplify the generating function into an expandable closed form.

Upon reviewing our entire research, which centers on the construction thought of generating functions, we systematically analyzed the construction process for specific forms and demonstrated its practical advantages through counting problems. Subsequently, by incorporating Euler's proof of the infinitude of prime numbers, we extended structural research on generating functions. Furthermore, through the analytical lens of generating functions applied to Bertrand-Chebyshev theorem, we highlighted that the absence of element contributions constrains the representational capacity of generating functions. Through in-depth analysis of original highly intricate sequence problems, we illustrated the potential of generating functions in addressing complex issues.

Our research provides a systematic analysis of the construction thought of generating functions. It is anticipated that with further exploration by other scholars into the construction thought of generating functions, they will unlock even greater potential in foundational mathematics and potentially in fields such as big data, artificial intelligence, and quantum computing. As a core tool, generating functions are expected to continue evolving, offering innovative ideas and fostering the deep integration of mathematics with real-world applications.

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