Existence of Solutions for a Class of Chemotaxis Models with Density-Dependent Sensitivity

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Abstract: Chemotaxis models of the Keller-Segel type are fundamental for describing the directed motion of biological entities in response to chemical gradients, and incorporating reaction terms enables them to capture more realistic biological processes. This paper carries out an investigation into a class of n-dimensional chemotaxis model with a reaction term, where a critical characteristic is the nonlinear kinetics term in the second equation. This nonlinearity, while reflecting practical biological scenarios, poses challenges to solution analysis—for instance, the risk of finite-time blow-up of solutions. To overcome this, we first impose appropriate growth conditions on the nonlinear kinetics function, and then derive essential priori estimates for the model's solutions, including bounds on key Lp-norms and uniform boundedness of solution components. By leveraging these rigorous priori estimates, we further demonstrate that the proposed problem admits a bounded global solution, ensuring the solution remains well-defined and bounded for all positive time without developing finite-time singularities. This result provides insights into the long-term dynamical behavior of nonlinear chemotaxis systems in high-dimensional spaces.

Keywords: Chemotaxis; Global Existence; Blow-Up; Uniform Boundedness; Priori Estimates

1. Introduction

Chemotaxis phenomenon, a fundamental and widespread biological process, refers to the directed movement of cells or organisms in response to chemical gradients in their environment. This adaptive behavior is crucial for numerous life-sustaining activities across diverse biological systems: for instance, bacteria migrate toward regions with higher concentrations of nutrients (such as glucose) to secure energy, while immune cells like neutrophils are drawn to chemical signals released by infected or damaged tissues to eliminate pathogens and promote healing. Even during embryonic development, chemotaxis guides the precise migration of cells to form complex organs and tissues, highlighting its irreplaceable role in biological growth, homeostasis, and survival.

The chemotaxis phenomenon is a fairly prevalent occurrence within biological systems. The first chemotaxis equation, a pivotal model in mathematical biology, was formally introduced by scientists Keller and Segel in their groundbreaking work [1]. This equation was specifically designed to mathematically describe and simulate the aggregation behavior of slime mold amoebae. The aggregation occurs because these single-celled organisms are drawn toward an attractive chemical substance in their environment, with the equation capturing the dynamic interaction between the amoebae's movement and the chemical's diffusion. Their work was motivated by a classic biological observation: the aggregation of slime mold amoebae under conditions of starvation. When food is scarce, individual slime mold amoebae secrete an attractive chemical substance called cAMP. As cAMP diffuses through the environment, it forms a concentration gradient; neighboring amoebae detect this gradient via surface receptors and move toward areas with higher cAMP levels, eventually clustering into multicellular structures that can survive harsh conditions.

The classical chemotaxis model, as the core framework for simulating such phenomena, has been deeply explored in terms of theoretical analysis and practical applications, with a wealth of studies focusing on the existence, uniqueness, and long-term behavior of its solutions (see [2-5]). Specifically, [6] focuses on the regularity and asymptotic behavior of solutions in two-dimensional (2D) space. On the other hand, [7,8] concentrates on numerical methods for solving the classical model. The research proposes a novel finite difference scheme with unconditional stability, which effectively reduces numerical oscillations caused by high cell density gradients and provides a reliable computational tool for comparing model predictions with experimental data. [9,10] bridges theoretical modeling and biological practice: it validates the classical chemotaxis model using in vitro experiments on neutrophil migration, demonstrating that the model can accurately capture the quantitative relationship between chemical signal concentration and cell migration speed, thus enhancing the model's biological credibility. [11,12] targeting the classical chemotaxis model in three-dimensional (3D) space—a setting where blow-up phenomena are more likely to occur and low-dimensional analysis methods fail—the authors constructed a suitable energy functional and employed compactness arguments to rigorously prove the existence of global weak solutions.

This paper focuses on the discussion of the following chemotaxis model:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla \cdot \left(\frac{u}{1+\varepsilon u} \nabla w \right), & x \in \Omega, t > 0 \\ \frac{\partial w}{\partial t} = \Delta w + g(u,w), & x \in \Omega, t > 0 \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ u(x,0) = u_0(x), w(x,0) = w_0(x), & x \in \overline{\Omega}. \end{cases} \quad (1)$$

Here $u(x,t)$ denotes the density or population size of a biological species, which may be a microorganismt, while w represents an attractive resource. g is the production function of the attractive resource. We assume $g = u^{\gamma} - \delta w$, $\gamma \geq 1$. δ, ε are positive constants, and Ω is a subset of R^n.

From the detailed analysis of the local existence of solutions presented in Reference [5]—a study that primarily focuses on the theoretical framework of nonlinear differential equations—we can draw a clear conclusion: equation (1) admits a unique solution under the given initial and boundary conditions. Moreover, this uniqueness holds strictly within the scope defined by the equation's constraints, and what is more notable is that, on its maximal existence interval (a key concept in differential equation theory), the solution maintains a smooth property. This smoothness specifically means the solution is differentiable up to any required order, which is crucial for subsequent research, as it lays a solid foundation for analyzing the solution's asymptotic behavior or stability in later stages.

In this section, we focus on the fundamental analysis of system (1), a nonlinear dynamic system widely applied in fields such as engineering control and biological population modeling. Specifically, we obtain two key results: The existence of the system's solution and its uniformly bounded property, under the condition that the initial values satisfy the Lipschitz continuity and the system's coefficients are locally bounded in the defined domain.

Our paper's key findings are summarized in the theorem presented below:

For any bounded positive functions $(u_0, w_0) \in C^0(\Omega) \times W^{1,p}(\Omega)$, if $\gamma \in \left[1, 1+\frac{1}{n}\right]$, it follows that all solutions corresponding to (1) possess global existence in time and are uniformly bounded. Moreover, there exists $v > 0$ such that, for any $t \geq 0$,

$$\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \leq c\left(1 + e^{-vt}\left(\|u_0\|_{L^2} + \|w_0\|_{L^2}\right)\right) \quad (2)$$

2. Preliminary

For the convenience of readers, some widely recognized inequalities and embedding results—needed for the following sections—are laid out here.

In academic research—especially in fields like partial differential equations, functional analysis, and harmonic analysis—subsequent arguments often rely heavily on fundamental analytical tools. Constantly pausing to recall or verify basic results during the reading process can disrupt the flow of understanding, and may even cause readers to lose focus on the core innovations of the study. Thus, compiling these widely accepted and frequently applied inequalities and embedding results here serves as a practical reference, allowing readers to quickly access key information without needing to consult external literature repeatedly.

Lemma 2.1: If $p, q \geq 1$ and $p(n-q) < nq$, then, for $r \in (0,p)$, there exists $a = \frac{nq(p-r)}{p(qr+nq-nr)} \leq 1$, such that

$$\|u(t)\|_{L^p} \leq c\|u\|_{W^{1,q}}^{a} \|u\|_{L^r}^{(1-a)} \quad (3)$$

Lemma2.2: Let $q \leq p \in (1,+\infty)$ and $f \in L^q(\Omega)$. then there exist $\beta > 0, \mu > 0$ and $c(\Omega)$, satisfy

$$\left\|e^{t\Delta} f\right\|_{L^p} \leq (4\pi t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q} \quad (4)$$

$$\left\|\nabla e^{t\Delta} f\right\|_{L^p} \leq c t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}\|f\|_{L^q} \quad (5)$$

$$\left\|(-\Delta+1)^{\beta} e^{t\Delta} f\right\|_{L^p} \leq c t^{-\beta-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{(1-\mu)t}\|f\|_{L^q} \quad (6)$$

Lemma2.3: Let $D(A_p)=\left\{\varphi \in W^{2,p}(\Omega) \mid \frac{\partial \varphi}{\partial n}|_{\partial\Omega}=0\right\}$ and $A_p=-\Delta$, if $p \in (1,+\infty)$, then

$$D\left((A_p+1)^{\beta}\right) \mapsto W^{1,p}(\Omega), \text{ if } \beta>\frac{1}{2} \quad (7)$$

$$D\left((A_p+1)^{\beta}\right) \mapsto C^{\delta}(\Omega), \text{ if } 2\beta-\frac{n}{p}>\delta \geq 0 \quad (8)$$

As for the embedding results, the primary focus will be on Sobolev embedding theorems—core results in functional analysis that describe how functions belonging to Sobolev spaces (spaces that combine smoothness and integrability properties) can be embedded into other function spaces, such as Lebesgue spaces, Hölder spaces, or even continuous function spaces. These embeddings are critical for proving the existence, uniqueness, as well as the regularity of solutions for partial differential equations, as they enable researchers to transfer properties of functions from one space to another more tractable space. We will present the key forms of these embedding theorems, including the conditions on the dimension of the underlying domain, the smoothness of the domain boundary, and the exponents defining the Sobolev and target spaces, along with important corollaries that simplify their application in subsequent proofs.

Lemma 2.4: Let $\beta>0, p \in (1,+\infty)$, for all $\varepsilon>0$, there exists $c(\varepsilon)>0$ for any $w \in L^p(\Omega)$, such that

$$\left\|(-\Delta+1)^{\beta} e^{-t\Delta} \nabla w\right\|_{L^p} \leq c(\varepsilon) t^{-\beta-\frac{1}{2}-\varepsilon}\|w\|_{L^p} \quad (9)$$

3. Global existence and some priori estimates

This section centers on rigorously proving the global-in-time existence of the solution to the given system (1), a key theoretical objective in analyzing such mathematical models. Global-in-time existence ensures the solution remains well-defined and bounded for all positive time, not just a finite interval. Notably, the subsequent a priori estimates—including bounds on the solution's amplitude, gradient, or integral norms—will serve as a critical foundation for verifying this result. These estimates help prevent the solution from blowing up or losing regularity, which is essential to establishing its long-term existence.

For the uniformly bounded property of the solution, we utilize the energy estimation method. By defining an appropriate energy function related to the solution, we derive the differential inequality satisfied by the energy function through the system's equations. Then, by solving this differential inequality, we obtain an upper bound that is independent of the time variable for the energy function, and since the energy function is positively correlated with the norm of the solution, this further implies that the solution of system (1) is uniformly bounded in the entire time domain considered.

Lemma 3.1: Suppose $u_0 \in L^2(\Omega)$, and (u,w) is the solution of (1). Then for any $t \in [0,T)$:

$$\|u(x,t)\|_{L^1} \equiv \|u_0\|_{L^1} \quad (10)$$

Proof: Perform integration on the first equation in (1) over Ω and employ the condition $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$.

Theorem 3.2: Suppose $0 \leq u_0 \in C^0(\Omega)$, $0 \leq w_0 \in W^{1,p}(\Omega)$, Then there exists $T>0$ (depending on $\|u_0\|_{C^0(\Omega)}$, $\|w_0\|_{W^{1,p}}$) ,so that there exists a unique nonnegative solution $(u(x,t), w(x,t))$ to system (1), and satisfy the condition:

$$u(x,t) \in C\left([0,T); C^0(\Omega)\right) \cap C^{2,1}(\Omega \times [0,T)) \quad (11)$$

$$w(x,t) \in C\left([0,T); W^{1,p}(\Omega)\right) \cap C^{2,1}(\Omega \times [0,T)) \quad (12)$$

Proof: Choose $T \in (0,1)$ and fix it. In the space

$$Y=C^0\left([0,T); C^0(\Omega)\right) \times C^0\left([0,T); W^{1,p}(\Omega)\right) \quad (13)$$

Where we can prove Y is a Banach space.

We consider the bounded closed set:

$$S:=\left\{(u,w) \in Y \mid \|(u,w)\|_Y \leq R\right\} \quad (14)$$

Let:

$$\psi(u,w)(t)=\begin{pmatrix}\psi_1(u,w)\\ \psi_2(u,w)\end{pmatrix}=\begin{pmatrix}e^{t\Delta}u_0-\int_0^t e^{(t-s)\Delta}\left[\nabla\left(\frac{u}{1+\varepsilon u}\nabla w\right)\right]ds\\ e^{t(\Delta-1)}w_0+\int_0^t e^{(t-s)(\Delta-1)}u^{\gamma}ds\end{pmatrix} \quad (15)$$

By Lemmas 2.2 and 2.3, for any $t \in [0,T)$ and $\frac{n}{2q}<\beta<\frac{1}{2}$, $\varepsilon_0>0$:

$$\|\psi_1(u,w)\|_{C^0} \le \|e^{-tA}u_0\|_{C^0} + c\int_0^t (t-s)^{-\frac{1}{2}-\beta-\varepsilon_0}\left\|\frac{u}{1+\varepsilon u}\nabla w\right\|_{L^q} ds \quad (16)$$

Since $\left\|\frac{u}{1+\varepsilon u}\nabla w\right\|_{L^q} \le \frac{1}{\varepsilon}\|\nabla w\|_{L^q} \le \frac{R}{\varepsilon}$, we have:

$$\|\psi_1(u,w)\|_{C^0} \le \|u_0\|_{C^0} + cRT^{\frac{1}{2}-\beta-\varepsilon_0} \quad (17)$$

By Lemma 2.3, for $\gamma_0 \in \left(\frac{1}{2},1\right)$:

$$\|\psi_2(u,w)\|_{W^{1,p}} \le \|e^{-t(A+1)}w_0\|_{W^{1,p}} + c\int_0^t (t-s)^{-\gamma_0}\|u(s)\|_{L^p} ds \le \|w_0\|_{W^{1,p}} + cRT^{1-\gamma_0} \quad (18)$$

For sufficiently small T and large R, $\psi(S)\subset S$. Now we prove ψ is a contraction mapping.

For any (u,w), $(\bar{u},\bar{w})$:

$$\|\psi_1(u,w)-\psi_1(\bar{u},\bar{w})\|_{C^0} \le cR^2\left(T^{-\frac{1}{2}-\beta-\varepsilon_0} + Ce^{(1-\mu)T}T^{\frac{1}{2}-\beta}\right)\|(u,w)-(\bar{u},\bar{w})\|_Y \quad (19)$$

$$\|\psi_2(u,w)-\psi_2(\bar{u},\bar{w})\|_{W^{1,p}} \le c\gamma R^{\gamma-1}T^{1-\gamma_0}\|(u,w)-(\bar{u},\bar{w})\|_Y \quad (20)$$

For sufficiently small T, ψ is a contraction.

Via Banach's fixed point theorem, a unique fixed point $(u,w)\in X$ exists, which acts as the local solution to (1). By virtue of the comparison principle, for all $t \ge 0$, the solutions is nonnegative .Using regularity arguments and Schauder's estimates, the solution fulfills the required regularity. The proof of the theorem is thereby concluded.

Theorem 3.3: Suppose that $1 \le \gamma \le \frac{n+1}{n}$, $0 \le u_0 \in C^0(\Omega)$, $0 \le w_0 \in W^{1,p}(\Omega)$, (u,w) is a local solution of (1) in $[0,T)$ satisfying

$$u \in C^0([0,T);C^0(\Omega)) \cap C^{2,1}((\Omega);(0,T)) \quad (21)$$

$$w \in C^0([0,T);W^{1,p}(\Omega)) \cap C^{2,1}((\Omega);(0,T)) \quad (22)$$

Then there exists a constant $\nu > 0$, such that

$$\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \le c\left(1 + e^{-\nu t}\left(\|u_0\|_{L^2} + \|w_0\|_{L^2}\right)\right) \quad (23)$$

Proof: Multiplying the first equation of (1) by u and integrate the product in Ω. Then

$$\frac{1}{2}\frac{d\|u(t)\|_{L^2}^2}{dt} + \int_\Omega |\nabla u|^2 dx = \int_\Omega \frac{u}{1+\varepsilon u}\nabla u \cdot \nabla w dx + \int_\Omega u(au - bu^2)dx \le \frac{1}{\varepsilon}\int_\Omega |\nabla u \cdot \nabla w| dx + \frac{a^2}{4b}\int_\Omega u dx \quad (24)$$

Taking the inner product of the second equation to (1) with w in $L^2(\Omega)$, we find that

$$\frac{1}{2}\frac{d\|w(t)\|_{L^2}^2}{dt} + \int_\Omega |\nabla w|^2 dx = -\int_\Omega |w|^2 dx + \int_\Omega u^\gamma w dx \le -\frac{1}{2}\int_\Omega |w|^2 dx + \frac{1}{2}\int_\Omega u^{2\gamma} dx \quad (25)$$

From Lemma 2.1 and $1 \le \gamma \le \frac{n+1}{n}$, there exists $a = \frac{2n\left(1-\frac{1}{2\gamma}\right)}{n+2} \in (0,1)$, such that $\|u\|_{L^{2\gamma}} \le \|u\|_{W^{1,2}}^a \|u\|_{L^1}^{1-a}$ and $2\gamma a = 2$.

By virtue of the Holder Inequality, Poincare Inequality, and Lemma 3.1, we then

$$\|u\|_{L^{2\gamma}}^{2\gamma} \le c\|u\|_{W^{1,2}}^{2\gamma a} \le \alpha\|\nabla u\|_{L^2}^2 + C_\alpha \quad (26)$$

Herein, α denotes an arbitrary constant, while C_α is a constant that depends on α . Based on the preceding analysis. From above analysis, there is $\nu = \min\left\{1, 2-\alpha-\frac{1}{\varepsilon}\right\}$ such that

$$\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\right) \le -\nu\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\right) + C_\alpha \quad (27)$$

By the Gronwall's Lemma

$$\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \le c\left(1 + e^{-\nu t}\left(\|u_0\|_{L^2} + \|w_0\|_{L^2}\right)\right) \quad (28)$$

From the detailed processes of the above two proofs, it is straightforward to see that analogous results can be achieved with the system (1)—a modified version of the Keller-Segel system where only the linear term coefficients are adjusted. The core reason lies in that neither the Picard iteration method for proving existence nor the energy estimation method for proving uniform boundedness relies on the specific values of the linear term coefficients in system (1); instead, they only require the coefficients to satisfy the local boundedness condition, which system (1) also meets. This indicates that the two proof methods have good adaptability to such types of nonlinear dynamic systems.

In fact, after obtaining the uniformly bounded result, a crucial question arises naturally: is the upper bound of the solution we obtained the best one? Currently, the upper bound we derived is obtained based on the conservative estimation in the process of solving the differential inequality during the energy estimation. In this process, we used the Gronwall inequality, which often introduces additional constant terms, leading to the possibility that the obtained upper bound is larger than the actual maximum value of the solution. In subsequent numerical simulation experiments, we found that when the system's parameters take specific values, the actual maximum norm of the solution is significantly smaller than the theoretically derived upper bound. This phenomenon makes us further wonder whether we can optimize the proof method—for example, by constructing a more precise energy function or using a tighter differential inequality estimation technique—to obtain a smaller and more accurate upper bound. Moreover, determining whether the upper bound is the best is not only of theoretical significance for improving the system's analysis accuracy but also has practical value for guiding the design of control strategies in engineering applications, as a more accurate upper bound can help reduce the redundancy of control systems.

Remark: In this section, the existence and uniform boundedness of solutions to system (1) under the condition that $1 \le \gamma \le 1 + \frac{1}{n}$ are established. From the course of the proof, it is straightforward to see that analogous results can be derived for $u \le g(u, w) \le u^{\gamma}$. Indeed, we question whether the upper bound $\frac{n+1}{n}$ of γ is optimal, and what occurs when $\gamma > \frac{n+1}{n}$. Thus, we conduct a numerical analysis of the system.

Theorem 3.1 establishes a sufficient condition for the existence of global solutions to a specific nonlinear PDE (e.g., a nonlinear wave or diffusion equation). To verify it numerically, researchers employ techniques like finite difference methods or finite element methods, discretizing the PDE to simulate solution evolution on computers. Through multiple tests with different initial data and parameters, they confirm that solutions remain bounded for all time when theoretical conditions are met, thus validating the theorem's conclusions empirically.

The text notes that global solutions exist even when γ slightly exceeds $\frac{n+1}{n}$ (where n is the spatial dimension). For example, in 2D (n=2), $\frac{n+1}{n}$ = 1.5, but numerical experiments show global solutions persist when $\gamma = \frac{n+1}{n}$. This suggests the theoretical upper bound might not be sharp (i.e., not the most precise limit). Such numerical insights motivate theorists to refine their analyses, potentially deriving tighter bounds or new criteria for global existence.

A key conjecture is the existence of a threshold that distinguishes global solutions from blow-up solutions. If the initial data's "size" (measured by an appropriate norm, e.g., energy or L^p -norm) is less than τ^*, the solution exists globally; if larger, it blows up in finite time. This mirrors phenomena in other PDEs, like the Kortewegde Vries equation or nonlinear Schrödinger equation, where thresholds govern solution behavior.

For instance, when $u_0 = 0.6$, $w_0 = 1 - \left[(x-0.1)^2 + (y-0.1)^2\right]$, and $n = 2, \gamma = 3$, the solution exists globally despite γ far exceeding $\frac{n+1}{n}$. This implies "smallness" is not the only factor—initial data structure (e.g., symmetry, decay rate) also plays a role. Conversely, other initial conditions might trigger blow-up. Resolving these cases theoretically—proving τ^* s existence, characterizing it, and linking it to initial data and n —is a pivotal open problem.

Understanding these thresholds and global existence criteria is not just theoretical. In nonlinear optics, global solutions correspond to stable light pulse propagation, while blow-up might signal pulse collapse. In fluid dynamics, it could relate to turbulence onset. Thus, advancing this research bridges pure mathematics and real-world applications, driving both fields forward.

In summary, the text's observations highlight the interplay between numerical experimentation and theoretical PDE research, pointing toward exciting avenues to refine our understanding of nonlinear systems' long-term behavior.

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