Application of Normal Inverse Gaussian Process in European Option Pricing

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Abstract: The traditional Black-Scholes (BS) model, based on the geometric Brownian motion assumption, struggles to account for the fat-tail, spike, and volatility time-varying characteristics of asset prices in financial markets, leading to pricing deviations in options. This paper introduces the Normal Inverse Gaussian (NIG) process into European option pricing research, verifying its applicability through theoretical derivations and empirical analysis. First, we review the core theories of Levy process, inverse Gaussian process, and NIG process, deriving the European option pricing formula under NIG. Second, using daily data from 5 U.S. listed companies and 3 global stock indices, we compare the fitting performance of NIG distribution with normal distribution. Finally, employing Monte Carlo simulation, we calculate S&P 500 European call option prices using both NIG and BS models, evaluating pricing accuracy through Mean Absolute Error Rate (MAER). Empirical results demonstrate that NIG distribution effectively captures the fat-tail characteristics of financial data, yielding pricing results closer to actual market prices with significantly lower MAER than BS models, providing a superior methodological choice for European option pricing.

Keywords:Normal Inverse Gaussian Process; European Option Pricing; Black-Scholes Model

1. Introduction

Research Background and Significance As a pivotal financial derivative, options have evolved into a cornerstone for risk management, asset allocation, and return enhancement since their debut on U.S. stock exchanges in the late 18th century. The pricing mechanism of European options—whose exercise is strictly limited to the expiration date—has become a focal point in financial mathematics. In 1973, Black and Scholes developed the seminal Black-Scholes (BS) model based on geometric Brownian motion assumptions, establishing theoretical foundations for option pricing and propelling the derivatives market's rapid expansion. However, as financial markets grew increasingly complex, the BS model's limitations became apparent. Real-world asset prices deviate from the idealized geometric Brownian motion, exhibiting pronounced fat-tail characteristics: returns peak higher than normal distributions, and extreme events occur more frequently. Moreover, asset prices demonstrate sudden jumps and volatility with distinct time-varying patterns—features that fundamentally contradict the BS model's core assumptions. Continuing to apply the BS model would lead to biased risk assessments and distorted pricing, potentially causing investment losses or creating arbitrage opportunities. The Levy process, as a stochastic process with independent stationary increments, provides a robust framework for modeling complex financial market dynamics. As a key branch of Levy processes, the Normal Inverse Gaussian (NIG) process exhibits infinite activity jumps. Its distribution flexibly fits spike-thick-tail data and can be constructed through time-transformed Brownian motion of inverse Gaussian processes, combining theoretical rigor with empirical applicability. Therefore, applying NIG processes to European option pricing holds significant theoretical and practical value for improving pricing accuracy and refining the derivatives pricing framework.

Research Status at Home and Abroad Foreign scholars have long focused on the shortcomings of traditional pricing models and explored improvement paths. Merton (1976) introduced a random jump term into the Black-Scholes (BS) model, proposing the jump-diffusion model, which partially addressed the neglect of price jumps. Heston (1993) constructed a stochastic volatility model, relaxing the assumption of constant volatility. Barndorff-Nielsen et al.

(1998) first proposed the Non-Intrinsically Gaussian (NIG) process and proved its infinite divisibility as a Levy process, providing a new perspective for financial data modeling. Rathgeber et al. (2019) empirically demonstrated that the Levy process effectively addresses two major flaws of the BS model, with the NIG process showing particular suitability for describing stock returns. Domestic research primarily focuses on empirical testing and extended applications of established models. Some scholars found that the NIG model outperforms other Levy processes in fitting financial data through comparative pricing effects, though most studies remain limited to single markets or small asset pools, lacking systematic multi-asset comparative analyses. Additionally, research on numerical computation methods and parameter estimation optimization for the NIG process requires further development, with no comprehensive practical framework yet established. Research Content and Framework The core research of this paper comprises three components: First, systematically analyzing the theoretical foundations of the NIG process to derive the European option pricing formula; Second, conducting empirical analysis to validate the NIG distribution's fit to financial data; Third, comparing the pricing accuracy of European options between the NIG model and the BS model. The research structure is as follows: Part I (Introduction) outlines the research background, significance, and current domestic and international research status; Part II (Theoretical Foundation) introduces the core definitions and properties of the Levy process, inverse Gaussian process, and NIG process; Part III (Option Pricing Formula Derivation) separately examines the pricing logic of the BS model and NIG model; Part IV (Empirical Research) includes data selection, comparison of fitting effects, and pricing accuracy verification; Part V (Conclusions and Outlook) summarizes the findings and future research directions.The research results are summarized, and the limitations and future research directions are pointed out.

2. Relevant Theoretical Basis

The Levy process is a class of continuous-time stochastic processes characterized by independent stationary increments and right-continuous, left-continuous paths, with the property $P(X_0=0)=1$. Its strict definition is as follows: Let $X=(X_t)_{t}\geq_0$ be a real-valued stochastic process. If it satisfies: (1) For any $n\geq1$ and $0\leq t_0<t_1<...<t_n$, the random variables X_{t0}, $X_{t1}-X_{t0}$,..., $X_{tn}-X_{tn-1}$ are mutually independent (independence incrementality); (2) For any s, $t\geq0$, $X_{t+s}-X_t$ and X_s are identically distributed (stationary incrementality); (3) The path $t\rightarrow X_t$ is right-continuous with probability 1 and has a left limit, then X is called a Levy process. There exists a one-to-one correspondence between Levy processes and infinitely divisible distributions: If a random variable Y has an infinitely divisible distribution—meaning for any $m\geq1$, there exist independent and identically distributed random variables $Y_1^{(m)}$,..., $Y_m^{(m)}$ such that $Y=Y_1^{(m)}+...+Y_m^{(m)}$, then a Levy process with increment distribution Y exists. Common distributions like normal and Poisson are infinitely divisible, which provides the foundation for the financial applications of Levy processes. The Levy-Khintchine formula reveals the characteristic exponential structure of the Levy process: $\psi(u) = i\gamma u- (\sigma^2u^2)/2 + \int(\mathbb{R}\backslash\backslash\{0\backslash\backslash\})[\exp(iu\ x) -1-iu\ x\ I(|x|<1)]v(dx)$, where (γ, σ^2, v) is the Levy triplet, $\gamma\in\mathbb{R}$ is the drift term, $\sigma^2\geq0$ is the intensity of the Brownian motion, and v is the Levy measure describing the frequency and amplitude distribution of jumps.

The inverse Gaussian (IG) process serves as the foundation of the Normal Inverse Gaussian (NIG) process, with its core concept being the characterization of the time distribution for the first arrival of a Brownian motion with positive drift at a positive level. Let W_s denote a standard Brownian motion, and $W_s + bs$ (b>0) represent the drift-modified Brownian motion. The time T(a,b) when this process first reaches the positive level a (a>0) follows an inverse Gaussian distribution IG(a,b), characterized by the probability density function $\varphi\_IG(u;a,b) = \exp(-a(\sqrt{}(-2iu + b^2) -b))$. The inverse Gaussian process $X^{(IG)} = (X_t^{(IG)})_t\geq_0$ is defined as a random process satisfying independent stationary increments, where the increments $X_t^{(IG)} -X_s^{(IG)} \sim IG(a(t-s), b)$, with $X_0^{(IG)} = 0$. Its probability density function is $f\_IG(x;a,b) = (a/\sqrt{}(2\pi x^3)) \exp(-((b^2x-2ab\sqrt{}x + a^2)/(2x)))$ (x>0), featuring a mean of a/b, variance of a/b^3, skewness of $3/(\sqrt{}(ab))$, and kurtosis o $3(1 + 5/(ab))$, exhibiting right-skewed characteristics suitable for modeling non-negative increments. (III) Normal Inverse Gaussian Process The NIG process is constructed through a time transformation of the drift-modified Brownian motion using the inverse Gaussian process. Specifically, let W_t denote a standard Brownian motion, and I_t be the inverse Gaussian process with parameters a=1 and b=δ√(α²-β²) (α> Given parameters α, β, δ> 0, where W and I are independent, the NIG process $X^{(NIG)} = (X_t^{(NIG)})_{t\geq 0}$ can be expressed as $X_t^{(NIG)} = \beta\delta^2 I_t + \delta W(I_t)$. Its characteristic function φ_NIG(u; α, β, δ) = exp(-δ(√(α² - (β + iu)²) - √(α² - β²))) exhibits a semi-thick-tailed distribution. As x → ±∞, the probability density function f_NIG(x; α, β, δ) ~ |x|^(−3/2) exp((√α + β)x), effectively capturing extreme value characteristics in financial data. The NIG distribution has a mean of δβ/√(α² - β²) and variance of δα²/(α² - β²)^(3/2). By adjusting parameters α, β, and δ, it can flexibly adapt to yield distributions of different financial assets. ## III. Derivation of European Option Pricing Formula ### (1) Black-Scholes Pricing Formula The BS model's core assumptions include: no transaction costs or taxes, constant risk-free interest rate, non-dividend-paying underlying asset, and geometric Brownian motion of asset price. The asset price S_t satisfies the stochastic differential equation: $dS_t/S_t = rdt + \sigma dW_t$, where r is the risk-free interest rate, σ is the volatility, and W_t is the standard Brownian motion. By applying Ito's lemma to transform the option price $f(S_t, t)$ and construct a risk-free hedging portfolio, we derive the Black-Scholes differential equation: ∂f/∂t + rS∂f/∂S + (1/2)σ²S²∂²f/∂S² = rf. Combined with the boundary condition for European call options f(S, T) = max(S-K, 0) (where K is the strike price and T is the expiration date), the BS pricing formula is obtained: $c_t = S_t\Phi(d_1)$ -Ke^(−r(T−t))Φ(d_2), where $d_1 = [\ln(S_t/K) + (r + \sigma^2/2)(T-t)] / [\sigma\sqrt{(T-t)}]$ and $d_2 = d_1 - \sigma\sqrt{(T-t)}$, with Φ(·) representing the cumulative distribution function of the standard normal distribution. The pricing formula for European put options can be derived through the call-put parity formula: p_t = Ke^(−r(T−t))Φ(−d_2) − $S_t\Phi(-d_1)$. ### (II) Pricing Formula under the Normal Inverse Gaussian Process (NIG) Model Under the NIG model, the underlying asset price process is defined as $S_t = S_0\exp(Z_t)$, where Z_t follows a NIG process with independent and stationary increments. As the NIG model is an incomplete market model, an equivalent martingale measure must be constructed via Esscher transformation to achieve risk-neutral pricing. The core idea of Esscher transformation is to adjust the probability measure so that the discounted asset price process becomes a martingale. Let the probability density function of Z_t be f_NIG(x; α, β, δ, μ). The Esscher-transformed density function is given by f_θ(x) = f_NIG(x; α, β, δ, μ) exp(θx)/M_NIG(θ), where M_NIG(θ) is the moment generating function and θ is the transformation parameter. To satisfy the risk-neutral condition E[e^(-rT)S_T] = S_0, the parameter θ must satisfy the equation: r = μ + δ(√(α² - (β+θ)²) - √(α² - (β+θ+1)²)). In this case, the Esscher-transformed Z_t still follows a NIG distribution with parameters (α, β+θ, δ, μ). Based on the risk-neutral pricing principle, the price of a European call option at time t is the discounted expectation of the maturity payoff: c_t = e^(-r(T-t))E_θ[(S_T-K)^+ | F_t]. Substituting S_T = S_t exp(Z_T-Z_t) and utilizing the NIG distribution properties, the pricing formula can be expressed as: c_t = S_t ∫(k^+∞) f_NIG(x; α, β+θ+1, (T-t)δ, (T-t)μ) dx-Ke^(-r(T-t)) ∫(k^+∞) f_NIG(x; α, β+θ, (T-t)δ, (T-t)μ) dx, where k = ln(K/S_t).Since the formula has no analytical closed-form solution, numerical methods are required for calculation. This paper adopts Monte Carlo simulation to achieve pricing.

3. Empirical Studies

Data Sources and Processing This study selects 5 U.S. listed companies (Ford Motor NYSE: F, Qualcomm NASDAQ: QCOM, 3M NYSE: MMM, Apple NASDAQ: AAPL, Amazon NASDAQ: AMZN) and 3 global stock indices (Shanghai Composite Index, FTSE 100 Index, S&P 500 Index) as research samples, covering the period from January 1, 2017 to August 31,2021, with a total of 1,134 daily observations. To eliminate non-stationarity in price series, daily logarithmic returns are calculated as $Z_t = \ln(S_t) - \ln(S_{t-1})$, where S_t represents the closing price on day t. Logarithmic returns not only possess better statistical properties but also satisfy time additivity, making them suitable for distribution fitting and volatility analysis.

3.2 Empirical Analysis of Option Pricing 1. Sample Selection and Parameter Setting The S&P 500 European Call Options were selected as the pricing benchmark due to their high trading activity and fairer market pricing. Two groups of options with different expiration dates were chosen: one expiring on September 13,2021 (3 days remaining), and another expiring on October 1, 2021 (20 days remaining), both containing different strike prices. Parameter settings are as follows: The risk-free rate r is set at the 3-month U.S. Treasury yield (0.05% annualized); the underlying asset's initial price S_0 is the S&P 500 closing price on the pricing benchmark date; the NIG model parameters α, β, δ, and μ were estimated using maximum likelihood based on the S&P 500's logarithmic return data; the BS model volatility σ was calculated using historical volatility estimation, specifically the annualized standard deviation of returns over the past 60 trading days.

Monte Carlo Simulation Steps (1) BS Model Simulation Steps: ① Set the time step $\Delta t = T/N$ (N=100, i.e., 100 time steps per period); ② Generate standard normal distribution random numbers Z, and iteratively calculate the asset price S_T at maturity using the asset price formula $S_{t+}\Delta t = S_t \exp((r- \sigma^2/2)\Delta t + \sigma Z\sqrt{\Delta t})$; ③ Calculate the option's maturity payoff max(S_T-K, 0) and discount it at the risk-free rate; ④ Repeat the simulation 10,000 times, and take the average of discounted returns as the option pricing result. (2) NIG Model Simulation Steps: ① Based on estimated NIG parameters, generate inverse Gaussian process random numbers through the Michael-Schumacher-Hass algorithm, and combine them with Brownian motion random numbers to construct NIG process increments; ② Calculate the asset price at maturity using the asset price formula $S_t = S_0 \exp(r + \delta(\sqrt{(\alpha^2 - (\beta+1)^2)} - \sqrt{(\alpha^2 - \beta^2)})t + X_t^{(NIG)})$; ③ Subsequent steps follow the BS model, with pricing results obtained through averaging 10,000 simulations. 3. Pricing Results and Accuracy Analysis The pricing accuracy is evaluated using the Mean Absolute Error Rate (MAER) formula: $MAER = (1/M)\Sigma|(|C_i - C_i|/C_i|)$, where C_i represents the model pricing result, C_i denotes the market price, and M is the sample size. A smaller MAER indicates higher pricing accuracy of the model. Pricing results show that the NIG model's pricing curve fits the market price curve significantly better than the BS model. For options with a remaining maturity of 3 days, the NIG model's MAER is 0.0417 compared to the BS model's MAER of 0.6634; for options with a remaining maturity of 20 days, the NIG model's MAER is 0.0297 compared to the BS model's MAER of 0.2948. Whether for short-term or medium-term options, the NIG model demonstrates substantially better pricing accuracy than the BS model, validating its advantage in capturing dynamic market characteristics.

4. Conclusions and Prospects

Research Conclusions This study systematically investigates the application of the NIG process in European option pricing. Through theoretical derivations and empirical analysis, the following conclusions are drawn: 1. As a significant extension of the Levy process, the NIG process exhibits sharp peaks and fat tails in its distribution, effectively fitting the actual yield distributions of financial assets while addressing the limitations of normal distributions in capturing extreme values and peak characteristics. Empirical results demonstrate that the NIG distribution shows superior fit to the yield distributions of all eight sample assets. 2. The European option pricing formula based on the NIG process achieves risk-neutral pricing through Esscher transformation, combined with Monte Carlo simulations to accurately calculate option prices. Compared to the traditional Black-Scholes (BS) model, the NIG model produces pricing results closer to actual market prices, with significantly reduced average absolute error rates—particularly demonstrating advantages in short-term option pricing. 3. Constructed through inverse Gaussian process time transformation of Brownian motion, the NIG process retains the independent and stationary increment characteristics of the Levy process while flexibly capturing asset price jumps and volatility time-varying properties, providing a more reliable methodology for option pricing in complex financial markets.

Research Limitations This study presents the following limitations: 1. Parameter estimation methods require optimization. While the maximum likelihood method used in this paper for NIG model parameters is susceptible to initial value sensitivity and potential local optima, the Markov Chain Monte Carlo (MCMC) method—capable of providing more robust parameter estimates without numerical optimization of log-likelihood functions—has not been applied in this research. 2. Limited sample coverage. The empirical analysis exclusively examines European call options on the S&P 500 Index, excluding put options, exotic options, and other market index derivatives, which may introduce sampling bias.

Additionally, the sample only includes options with different strike prices, neglecting the pricing effects of the same option across different time points. 3. Computational efficiency needs improvement. Monte Carlo simulations require extensive iterative calculations, resulting in prolonged computation time, and the study does not incorporate variance reduction techniques to enhance simulation efficiency.

Conclusion: Future research could focus on the following aspects: 1. Optimize parameter estimation methods by introducing MCMC techniques to enhance the accuracy of NIG model parameters, and compare the effectiveness of different estimation approaches. Expand the sample scope to include put options, American options, and exotic options, covering stock indices and individual stock options from more countries and regions to strengthen the universality of conclusions. 3. Improve numerical computation methods by integrating control variable techniques and variance reduction methods like importance sampling to enhance the efficiency and accuracy of Monte Carlo simulations; simultaneously explore the application of analytical numerical methods such as fast Fourier transform in NIG option pricing. 4. Investigate the integration of NIG processes with other financial models, such as combining NIG processes with stochastic volatility models, to further improve the ability to characterize market dynamics and provide more comprehensive solutions for option pricing.

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