A Structure-Preserving Algorithm for the Fractional Klein-Gordon-Zakharov System

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Abstract: The fractional Klein-Gordon-Zakharov (FKGZ) system introduces non-local effects via fractional derivatives, serving as a key framework for modeling interactions between Langmuir and ion-acoustic waves in complex plasmas and other systems with long-range interactions. This work focuses on the design and analysis of an innovative structure-preserving algorithm tailored for numerically solving the FKGZ system. Unlike traditional methods that often suffer from significant energy drift and poor long-term fidelity, our approach strictly preserves the inherent Hamiltonian and multi-symplectic structures of the continuous model, ensuring accurate simulation of long-time nonlinear wave dynamics. We initiate our work by transforming the original FKGZ system into an equivalent infinite-dimensional Hamiltonian framework, thereby elucidating its multi-symplectic architecture and conserved invariants, such as energy and momentum. A key challenge addressed is the treatment of the fractional Riesz derivative, for which we employ a weighted and shifted Grünwald-Letnikov formula to achieve a high-order approximation while maintaining the structure of the discrete system. The central contribution of our research resides in developing a fully discrete scheme that integrates the discrete fractional derivative operator with a temporal implicit midpoint rule, which provides rigorous demonstrations that our proposed scheme maintains a discrete counterpart of the multi-symplectic conservation law and precisely upholds the total discrete energy and momentum. These preservation characteristics are vital for ensuring numerical stability and reliability in extended-time simulations. A series of numerical experiments is performed to substantiate the theoretical results. We verify the precision of our method via convergence analyses. Furthermore, we emulate the evolution of soliton solutions and the interplay among fractional-order wave packets across extended durations. Findings indicate that our structure-preserving algorithm surpasses conventional non-preserving techniques in robustness, accuracy of invariant conservation, and capacity to replicate authentic physical behaviors, particularly under conditions of pronounced nonlinearity and diverse fractional orders. This work provides a reliable and efficient numerical tool for exploring complex phenomena in fractional nonlinear wave dynamics. The methodology developed in this work represents a significant advancement in the numerical treatment of coupled fractional nonlinear systems. By addressing both the theoretical foundations and practical implementation challenges, we establish a comprehensive framework that can be adapted to other fractional partial differential equations with Hamiltonian structures. The demonstrated robustness of our algorithm across various fractional orders and nonlinear regimes suggests its potential as a general-purpose tool for investigating non-local wave phenomena in diverse physical contexts.

Keywords: Fractional Klein-Gordon-Zakharov Equation; Structure-Preserving Algorithm; Multi-Symplectic Integrator; Energy Conservation; Momentum Conservation, Riesz Derivative; Fractional Laplacian; Soliton Interaction; Implicit Midpoint Rule.

1. Introduction

1.1 Background and Motivation

The study of nonlinear wave phenomena is a

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cornerstone of modern mathematical physics, with applications spanning plasma physics, nonlinear optics, Bose-Einstein condensates, and biological systems. Among the numerous models formulated to depict these phenomena, the Klein-Gordon-Zakharov (KGZ) system occupies a distinguished position. Initially developed to represent the interaction between Langmuir waves (high-frequency electron oscillations) and ion-acoustic waves (low-frequency ion density variations) in plasmas [1], the KGZ system comprises a coupled set of nonlinear partial differential equations (PDEs). Its standard form is given by:

$$\begin{cases} u_{tt} - \Delta u + u + vu + |u|^2 u = 0 \\ v_{tt} - c^2 \Delta v - \Delta(|u|^2) = 0 \end{cases} \quad (1)$$

where $u(x,t)$ is a complex-valued function, $v(x,t)$ is a real-valued function, and c is the ion-acoustic speed. The cubic nonlinear term $|u|^2 u$ in the first equation represents the self-interaction of the Langmuir wave, while the term $\Delta(|u|^2)$ in the second equation acts as a driving force for the ion-acoustic wave, coupling the two fields nonlinearly.

The KGZ system is known to possess rich mathematical structures, including a Hamiltonian formulation, multi-symplecticity, and several conserved quantities such as energy and momentum. These attributes are not merely of mathematical interest; they underpin the fundamental physics of the model, dictating the long-term solution behavior, the stability of solitary waves, and the recurrence of initial configurations.

In recent decades, the emergence of fractional calculus has provided a potent framework for representing anomalous diffusion and non-local transport phenomena that classical integer-order derivatives cannot sufficiently characterize [2]. Fractional derivatives, featuring non-local kernels, are intrinsically appropriate for encapsulating memory effects and extensive spatial interactions. This progression has prompted the advancement of fractional partial differential equations (FPDEs) as superior models for systems in porous media, viscoelasticity, finance, and notably, in complex plasmas where particle interactions may deviate from the standard local picture.

The incorporation of fractional derivatives into physical models represents more than just a mathematical generalization; it reflects a fundamental shift in how we conceptualize spatial and temporal relationships in complex systems. Traditional local models assume that interactions occur primarily between immediate neighbors, but many physical systems exhibit long-range correlations that require non-local mathematical descriptions. Fractional calculus provides a natural language for expressing these non-local relationships, with the fractional order serving as a continuous parameter that interpolates between different regimes of interaction. This flexibility is particularly valuable in plasma physics, where the presence of external magnetic fields, non-Maxwellian particle distributions, and turbulent processes can create effective long-range interactions that are not captured by standard differential operators.

The fractional Klein-Gordon-Zakharov (FKGZ) equation emerges naturally from this context. By replacing the standard Laplacian Δ with a fractional Riesz derivative $(-\Delta)^{\frac{\alpha}{2}}$ (where $1 \le \alpha \le 2$), we introduce spatial non-locality into the wave propagation dynamics. The FKGZ system can be written as:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + u + vu + |u|^2 u = 0, x \in \Omega, t \in (0,T] \\ v_{tt} - c^2 (-\Delta)^{\frac{\beta}{2}} v - (-\Delta)^{\frac{\beta}{2}} (|u|^2) = 0, x \in \Omega, t \in (0,T] \end{cases} \quad (2)$$

Under the prescribed initial and boundary conditions, the parameters α and β control the degree of non-locality in the Langmuir and ion-acoustic wave components, respectively. Setting $\alpha = \beta = 2$ reduces the system to the classical KGZ model. The FKGZ system is thereby anticipated to deliver more accurate modeling of wave interactions in media exhibiting fractal characteristics or power-law correlation functions.

The analytical study of the FKGZ system is exceedingly difficult. Closed-form solutions are scarce, and the well-posedness theory for such nonlinear FPDEs is still under development. Consequently, numerical simulation becomes an indispensable tool for understanding the dynamics of the FKGZ system. However, the numerical solution of FPDEs is uniquely challenged by the non-locality of fractional derivatives, which gives rise to dense coefficient matrices and, consequently, prohibitively high computational and memory requirements. More

critically, standard numerical methods developed for classical PDEs, when applied naively to FPDEs, often fail to preserve the essential geometric structures and physical invariants of the continuous system. This failure can lead to numerical instabilities, spurious dissipation or amplification, and qualitatively incorrect long-time behavior, such as the artificial decay of solitary waves or the drift of conserved quantities.

The development of structure-preserving algorithms for FPDEs thus addresses a critical need in computational physics. Traditional numerical methods often introduce artificial dissipation or fail to maintain fundamental invariants over long time scales, leading to qualitatively incorrect solutions. This is particularly problematic when studying phenomena such as soliton interactions, recurrence behavior, or chaotic dynamics, where the accurate preservation of geometric structures is essential for capturing the true physical behavior. Structure-preserving methods, by contrast, are designed to inherit the geometric properties of the continuous system, ensuring that key physical quantities are maintained at the discrete level. This approach represents a paradigm shift in numerical analysis, where the design criteria extend beyond traditional measures of accuracy and stability to include fidelity to the underlying mathematical structure of the physical model.

1.2 Literature Review

Our work sits at the intersection of three active research areas: the fractional KGZ equation, numerical methods for FPDEs, and structure-preserving algorithms.

Firstly, the literature on the fractional KGZ system is relatively sparse compared to its classical counterpart. Most existing works focus on analytical aspects. While the local and global well-posedness in certain Sobolev spaces has been established in some studies [3], a separate line of research has investigated the existence of solitary wave solutions using variational methods. However, due to the complexity introduced by the fractional operators and the nonlinear coupling, a comprehensive dynamical understanding is lacking, creating a strong demand for reliable numerical simulations.

Recent analytical investigations have advanced the understanding of the FKGZ system's fundamental properties. Initial work by Guo and Huang established criteria for solution existence and blow-up. Building upon this foundation, Liu and Yin further elucidated the dynamics of solitary waves and related blow-up phenomena. These analytical works provide important theoretical foundations but also highlight the need for numerical tools to explore regimes where analytical methods are limited. In studying solitary wave solutions, the variational framework has proven to be a particularly productive framework, key to elucidating the connection between the fractional order and the stability of these localized structures. However, the time-dependent dynamics, especially in regimes of strong nonlinearity or non-integer fractional orders, remain largely unexplored territory that requires sophisticated numerical investigation.

Recently, several new results have emerged on structure-preserving algorithms for fractional PDEs, including the work of Li et al. (2024) on linearly implicit energy-preserving schemes for the FKGZ system and the multi-symplectic formulations proposed by Gorshkov and Dutykh (2022). Our work builds upon these advances by providing a fully discrete scheme that simultaneously preserves multi-symplecticity, energy, and momentum.

Secondly, significant research efforts have been devoted to the numerical approximation of fractional derivatives. Standard definitions such as Grünwald-Letnikov, Riemann-Liouville, and Caputo are frequently implemented using finite difference, finite element, or spectral methods. Of central importance to our work is the Riesz derivative, where finite difference schemes utilizing the weighted and shifted Grünwald-Letnikov (WSGL) formulas are widely adopted for their computational appeal and capacity to achieve higher-order accuracy. The main computational bottleneck is the dense structure of the resulting discretization matrices, which has spurred research into fast algorithms based on the fast Fourier transform (FFT) or hierarchical matrices to reduce the complexity from $O(N^2)$ to nearly $O(N \log N)$.

The evolution of numerical methods for fractional derivatives has witnessed several key advancements. Initial foundations were laid by Meerschaert and Tadjeran, whose finite difference techniques for advection-dispersion equations were widely adapted [4]. Subsequent research by Yang et al. tailored methods to the

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symmetric non-locality of the Riesz derivative. Recent advancements focus on high-order compact schemes, achieving superior accuracy while maintaining computational efficiency. Spectral methods have also shown promise for FPDEs, with exponential convergence rates demonstrated for certain classes of problems. However, these advanced discretization techniques have rarely been combined with structure-preserving temporal integrators, creating an opportunity for the development of more comprehensive algorithms that address both accuracy and structure preservation simultaneously.

Thirdly, the field of structure-preserving (or geometric) numerical integration focuses on designing methods that maintain the geometric properties of the original system, including symplecticity and conservation laws. A cornerstone of this field for Hamiltonian PDEs is the multi-symplectic integrator. The multi-symplectic integrator, originally developed by Bridges and Reich, provides a systematic framework for preserving local conservation laws in Hamiltonian PDEs [5]. These schemes conserve a discrete analogue of the multi-symplectic structure, which in turn ensures the global conservation of energy and momentum. This rationale has motivated the development of various structure-preserving methods for the classical KGZ system. These methods have demonstrated superior performance in long-time simulations and in capturing the correct interaction of solitons.

The theoretical foundations of structure-preserving algorithms are deeply rooted in geometric numerical integration, as comprehensively treated in the monographs by Hairer and Reich. For Hamiltonian ODEs, symplectic integrators have been shown to possess remarkable long-time stability properties, with errors in energy that typically remain bounded rather than growing secularly. The introduction of multi-symplectic integrators by Bridges and Reich extended this philosophical principle to Hamiltonian PDEs. This advancement spurred the creation of numerous tailored algorithms, exemplified by the linearly implicit conservative scheme for the classical KGZ system developed by Wang et al., which Cai and Wang later adapted to its fractional counterpart. However, these methods typically preserve only one or two invariants, whereas our approach aims to preserve the complete multi-symplectic structure along with all fundamental conservation laws (Antoine et al., 2013).

Integrating structure-preserving principles with the numerical solution of fractional PDEs presents a dual challenge: the non-local character of fractional derivatives often obscures the underlying Hamiltonian or multi-symplectic structures. Consequently, existing research remains in its infancy, with efforts largely confined to simpler fractional Hamiltonian ODEs and specific fractional Schrödinger equations. This landscape reveals a clear research gap: no structure-preserving algorithm currently exists for the coupled, nonlinear fractional KGZ system, a challenge this work directly addresses.

While structure-preserving algorithms have yielded remarkable success for integer-order partial differential equations, their applicability to fractional systems is severely constrained by challenges of a fundamentally different nature. The global memory effect of fractional derivatives disrupts the local geometric properties that traditional techniques rely upon, demanding innovative mathematical frameworks. Initial progress has been made through the contributions of Gorshkov and Dutykh, who devised multi-symplectic formulations for specific fractional nonlinear wave equations, and Hong and Liu, who demonstrated the practical viability of such methods for the fractional nonlinear Schrödinger equation [6]. Our work develops a structure-preserving scheme for the coupled fractional KGZ system. Here, the primary difficulty escalates due to the nonlinear coupling between two wave fields, each potentially associated with a distinct fractional order [7]. The resolution of this complex problem marks a substantial leap forward in the numerical analysis of fractional systems, establishing a new benchmark for structure-preserving discretization in this challenging domain.

1.3 Our Contributions

This paper aims to address this gap by proposing, analyzing, and validating a novel structure-preserving algorithm for the FKGZ system. This article makes the following key contributions:

(1) Hamiltonian Reformulation. We successfully reformulate the FKGZ system into an infinite-dimensional Hamiltonian form, clearly

identifying its Hamiltonian functional and the associated Poisson structure. Furthermore, we derive its multi-symplectic formulation, which unveils the local conservation laws of symplecticity.

(2) Development of a Fully Discrete Structure-Preserving Scheme. The specific steps undertaken in the scheme's construction are detailed as follows:

Employing a high-order WSGL formula for the spatial fractional Riesz derivatives.

Applying the implicit midpoint rule for time discretization.

We demonstrate that the proposed numerical formulation, through this specific combination, gives rise to a scheme which is multi-symplectic and simultaneously preserves the discrete total energy and momentum exactly, thus establishing its structure-preserving character.

(3) Rigorous Numerical Analysis. A rigorous numerical analysis is conducted for the proposed scheme, providing comprehensive proofs of its conservation properties, encompassing multi-symplecticity, energy, and momentum. We also discuss the solvability of the resulting nonlinear system and establish conditions for its unique solution.

(4) Comprehensive Numerical Validation. A series of numerical experiments is presented to validate the proposed algorithm's efficacy and showcase its conservation properties and convergence behavior. These include:

Convergence tests to verify the theoretical order of accuracy.

Long-time simulations to showcase the precise conservation of energy and momentum, contrasting it with the performance of non-preserving methods.

Simulations of soliton dynamics and wave interactions, illustrating the robustness of our method and the physical effects of varying the fractional order α.

Beyond these primary contributions, our work offers several secondary advancements that merit emphasis. First, we provide a comprehensive framework for handling the computational challenges associated with dense fractional derivative matrices, including strategies for efficient implementation and memory management. Second, we offer new insights into the physical behavior of the FKGZ system across different fractional orders, revealing how non-locality affects wave dispersion, soliton stability, and nonlinear interactions. Third, we establish connections between the mathematical structure of the FKGZ system and its physical manifestations, demonstrating how structure-preserving discretization leads to more physically realistic simulations. Finally, our implementation serves as a template that can be adapted to other fractional Hamiltonian systems, potentially influencing numerical approaches beyond the specific equation considered here.

2. Preliminaries and Model Problem

2.1 Fractional Calculus

In this subsection, we recall the essential definitions of fractional derivatives used in this work. Let $f(x)$ be a function defined on the real line R.

Definition 2.1 (Left and Right Riemann-Liouville Fractional Derivatives). For a function $f \in L^1(R)$, the left and right Riemann-Liouville fractional derivatives of order $\gamma > 0$ are respectively defined as follows:

$$ {}_{-\infty}D_x^{\gamma} f(x) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_{-\infty}^{x} \frac{f(\xi)}{(x-\xi)^{\gamma-n+1}} d\xi, \tag{3} $$

$$ {}_{x}D_{+\infty}^{\gamma} f(x) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_{x}^{+\infty} \frac{f(\xi)}{(x-\xi)^{\gamma-n+1}} d\xi, \tag{4} $$

with $\Gamma(\cdot)$ representing the Gamma function.

Definition 2.2 (Riesz Fractional Derivative). Building upon the left and right Riemann-Liouville derivatives, the Riesz fractional derivative of order γ is defined as their linear combination [8]:

$$ \frac{\partial^{\gamma}}{\partial |x|^{\gamma}} f(x) = -c_{\gamma} \left({}_{-\infty}D_x^{\gamma} f(x) + {}_{x}D_{+\infty}^{\gamma} f(x) \right) \tag{5} $$

where $c_{\gamma} = \frac{1}{2\cos(\frac{\pi\gamma}{2})}$, $\gamma \neq 1,3,5,...$

For γ =2, the Riesz derivative reduces to the standard negative Laplacian, $-\frac{\partial^2}{\partial x^2}$. In higher dimensions, the fractional Laplacian $(-\Delta)^{\frac{\gamma}{2}}$ is a generalization of the Riesz derivative.

The mathematical properties of fractional derivatives have profound implications for their physical interpretation and numerical treatment. Unlike integer-order derivatives, which are local operators, fractional derivatives incorporate information from the entire domain, with the strength of non-local interactions dictated by the power-law kernel. This non-locality manifests

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mathematically in the dense structure of discretization matrices and physically in anomalous transport phenomena. The Riesz derivative specifically represents a symmetric form of fractional differentiation that preserves certain important physical properties, such as rotational invariance in higher dimensions [9][10]. From a numerical perspective, the treatment of fractional derivatives requires careful consideration of boundary conditions and domain truncation, as the theoretical definitions involve integrals over infinite domains. In practical implementations, various strategies have been developed to address these issues, including domain truncation with appropriate decay assumptions, non-reflecting boundary conditions, and embedding techniques that minimize artifacts from finite computational domains.

2.2 The Fractional Klein-Gordon-Zakharov System

The primary objective of this work is to develop and analyze a structure-preserving numerical scheme for a class of fractional systems. To this end, we consider the following one-dimensional fractional Klein-Gordon-Zakharov system, defined on the bounded interval $\Omega = [a,b]$. For the purposes of this study, we impose periodic boundary conditions. This configuration is deliberately selected to achieve two key aims: to maintain simplicity and clarity in the discretization of the spatial domain, and, more importantly, to enable an unambiguous investigation into the inherent structure-preserving capabilities of the proposed algorithm.

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + u + vu + |u|^2 u = 0, x \in [a,b], t > 0 \\ v_{tt} - c^2(-\Delta)^{\frac{\beta}{2}} v - (-\Delta)^{\frac{\beta}{2}}(|u|^2) = 0, x \in [a,b], t > 0 \\ \mathrm{u}(\mathrm{x},0) = \mathrm{u}_0(x), \mathrm{u}_t(\mathrm{x},0) = \mathrm{u}_1(x) \\ \mathrm{v}(\mathrm{x},0) = \mathrm{v}_0(x), \mathrm{v}_t(\mathrm{x},0) = \mathrm{v}_1(x) \end{cases} \tag{6}$$

where $u(x,t)$ is a complex-valued function, $v(x,t)$ is a real-valued function, $c > 0$ is a constant, and $1 < \alpha, \beta \le 2$. The operators $(-\Delta)^{\frac{\alpha}{2}}$ and $(-\Delta)^{\frac{\beta}{2}}$ are the one-dimensional fractional Laplacians, which under periodic boundary conditions are equivalent to the Riesz derivative defined above. The nonlinear coupling term $u \cdot v$ in the first equation represents the interaction between the Langmuir wave envelope and the ion density fluctuation, which is the primary mechanism for energy exchange between the two wave components [11].

This system conserves several important quantities. The most crucial ones are the energy $E(t)$ and the momentum $M(t)$, given by:

$$E(t) = \int_\Omega [|u_t|^2 + |(-\Delta)^{\frac{\alpha}{4}} u|^2 + |u|^2 + \frac{1}{2} v|u|^2 + \frac{1}{2}|u|^4 + \frac{1}{2} v_t^2 + \frac{c^2}{2} |(-\Delta^{\frac{\beta}{4}}) v|^2] dx \tag{7}$$

$$M(t) = \mathrm{Im} \int_\Omega u_t \bar{u} dx + \int_\Omega v_t (-\Delta)^{\frac{\beta}{4}} v dx \tag{8}$$

The total energy functional $E(t)$ in Eq. (7) explicitly involves the fractional differential operator $(-\Delta)^{\frac{\alpha}{4}}$, which fully embodies the nonlocal effects introduced by fractional spatial derivatives. In Eq. (8), Im (·) denotes the imaginary part of the complex quantity.

In contrast, the momentum functional $M(t)$ in Eq. (8) only consists of integer-order first derivatives without any fractional derivative terms. This distinction arisesarises because momentum conservation is induced by spatial translation invariance and merely relies on first-order spatial gradients, while energy conservation simultaneously incorporates fractional-order spatial wave contributions.

A primary goal of our numerical method is to preserve discrete analogs of $E(t)$ and $M(t)$.

The physical interpretation of the FKGZ system extends beyond its original plasma physics context to various systems exhibiting wave interactions with non-local coupling [12]. The fractional orders α and β control the degree of non-locality in the Langmuir and ion-acoustic wave components, respectively. When $\alpha, \beta < 2$, the system exhibits superdiffusive behavior, with wave packets spreading more rapidly than in the classical case [20]. This has implications for energy transfer, pattern formation, and the stability of coherent structures. The conservation laws for energy and momentum remain valid for arbitrary fractional orders, reflecting fundamental physical principles that transcend the specific form of the differential operators [13]. From a mathematical perspective, the FKGZ system represents an interesting case study in coupled fractional nonlinear waves, with the potential for rich dynamics, including

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soliton solutions, blow-up phenomena, and turbulent cascades. Our numerical approach aims to capture this full range of behaviors while respecting the underlying physical constraints embodied in the conservation laws.

For periodic boundary conditions on a bounded interval $[a,b]$, the Riesz fractional derivative is equivalent to the fractional Laplacian, a property that greatly simplifies numerical implementation and ensures consistency with the continuous theory.

3. Hamiltonian and Multi-Symplectic Structures of the FKGZ System

To design a structure-preserving algorithm, one must first understand the geometric structure of the continuous model. A foundational aspect of our analysis lies in identifying and formulating the inherent Hamiltonian and multi-symplectic structures that govern the FKGZ system.

3.1 Hamiltonian Formulation

A strategic reformulation is employed, wherein the introduction of canonical variables transforms the second-order system in time into a first-order Hamiltonian form, thereby facilitating the subsequent structural analysis [14]. Let $p=\frac{\partial u}{\partial t}$ and $q=\frac{\partial v}{\partial t}$. The FKGZ system can then be rewritten as:

$$\begin{cases} u_t = p \\ p_t = -(-\Delta)^{\frac{\alpha}{2}}u - u - vu - |u|^2 u, \\ \mathrm{v}_t = q \\ q_t = c^2(-\Delta)^{\frac{\beta}{2}}v + (-\Delta)^{\frac{\beta}{2}}(|u|^2) \end{cases} \tag{9}$$

A fundamental observation is that the system possesses the canonical structure of an infinite-dimensional Hamiltonian system, which is expressed as:

$$\frac{\partial z}{\partial t} = J\frac{\delta H}{\delta z} \tag{10}$$

where $z=(u,p,v,q)^T$ is the state vector, $\frac{\delta H}{\delta z}$ is the variational derivative of the Hamiltonian functional H, and J is the symplectic operator given by:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{11}$$

The Hamiltonian functional $H[z]=E(t)$ is precisely the energy defined earlier:

$$H[u,p,v,q] = \int_\Omega [\frac{1}{2}|p|^2 + \frac{1}{2}|(-\Delta)^{\frac{\alpha}{4}}u|^2 + \frac{1}{2}|u|^2 + \frac{1}{2}v|u|^2 + \frac{1}{4}|u|^4 + \frac{1}{2}q^2 + \frac{c^2}{2}|(-\Delta)^{\frac{\beta}{4}}v|^2]dx \tag{12}$$

It is straightforward to verify that $\frac{\delta H}{\delta p}=p$, $\frac{\delta H}{\delta u}=(-\Delta)^{\frac{\alpha}{2}}u+u+vu+|u|^2u$, $\frac{\delta H}{\delta q}=q$, $\frac{\delta H}{\delta v}=-\frac{1}{2}|u|^2-c^2(-\Delta)^{\frac{\beta}{2}}v$. Substituting these into the Hamiltonian system $Z_t=J\frac{\delta H}{\delta z}$ recovers the first-order form of the FKGZ system.

Serving as a foundational framework, the Hamiltonian formulation enables a geometric interpretation of the FKGZ system's dynamics, elucidating its underlying conservation laws and symplectic flow [15][16]. The symplectic operator J defines a Poisson bracket structure that governs the evolution of observables through the equation $(F,G)=(\frac{\delta F}{\delta z},\frac{J\delta G}{\delta z})$ for functionals F and G. This bracket satisfies the Jacobi identity and antisymmetry, making it a genuine Poisson structure. The Hamiltonian itself generates the time evolution through $\frac{dF}{dt}=\{F,H\}$ for any functional F. This formulation immediately implies the conservation of energy, as $\{H,H\}=0$ identically. More generally, functionals that Poisson-commute with the Hamiltonian are conserved quantities.

3.2 Multi-symplectic Formulation

The multi-symplectic formulation provides a more detailed local description of the system's geometry. To derive it, we need to introduce potential functions for the fractional derivatives. This is non-trivial. Nevertheless, under periodic boundary conditions, it becomes feasible to leverage the analytical properties of the fractional Laplacian. We define auxiliary variables $\omega=(-\Delta)^{\frac{\alpha}{4}}u$ and $\phi=(-\Delta)^{\frac{\beta}{4}}v$. While a full canonical multi-symplectic formulation with standard differential forms is complicated by the non-local fractional operators, a semi-discrete

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multi-symplectic perspective can be presented, which approximates the continuous system by preserving a discrete symplectic structure in space.

The key idea is that the Hamiltonian structure implies the conservation of symplecticity. It is an established theoretical result that a numerical method which faithfully preserves the symplectic structure of the underlying Hamiltonian ODE system (obtained after spatial discretization) will inherently demonstrate excellent long-time stability and conservation properties, as it faithfully mirrors the geometric structure of the continuous phase flow [17][18][19]. To achieve this, our numerical scheme is constructed via a two-stage procedure. The continuous system is first reduced to a high-dimensional Hamiltonian ODE system through a structure-preserving discretization of the fractional spatial derivatives. This is followed by the application of a symplectic time-stepping method, ensuring the geometric properties of the Hamiltonian ODEs are conserved during temporal integration.

By decomposing a PDE into constituent symplectic structures in multiple directions, the multi-symplectic formalism delivers a nuanced understanding of its geometric foundations. In contrast to the well-established theory for classical PDEs, formulating a genuine multi-symplectic structure for fractional systems remains an open challenge, primarily due to the non-local character of fractional differential operators. However, a pragmatic approach can be adopted by considering the semi-discrete system obtained after spatial discretization. This system is a finite-dimensional Hamiltonian ODE that possesses a standard symplectic structure [20]. This semi-discrete perspective bridges the gap between the continuous fractional system and fully discrete numerical methods, providing a principled approach to structure preservation even when a complete multi-symplectic theory for the continuous fractional system remains elusive.

4. Construction of the Fully Discrete Structure-Preserving Scheme

In this section, we detail the construction of our fully discrete numerical scheme.

4.1 Spatial Discretization: Fractional Derivative Approximation

We discretize the spatial domain $[a,b]$ uniformly with grid points $x_j = a + jh$, for $j = 0,1,\ldots,N$, where $h = (b-a)/N$ is the spatial step size. We denote the numerical approximations as $U_j(t) \approx u(x_j,t)$, $V_j(t) \approx v(x_j,t)$ [21].

To approximate the Riesz fractional derivative $(-\Delta)^{\frac{\alpha}{2}} u(x_j)$, we use the weighted and shifted Grünwald-Letnikov (WSGL) formula [22][23]. For a sufficiently smooth function $u(x)$, the α-th order Riesz derivative can be approximated by:

$$(-\Delta)^{\frac{\alpha}{2}} u(x_j) \approx \frac{1}{h^\alpha} \sum_{k=0}^{N} g_k^{(\alpha)} u(x_{j-k}) \tag{13}$$

where $N = j$ for interior points, $N = \min(j,K)$ for boundary truncation with sufficiently large K, and K is a sufficiently large truncation parameter chosen to achieve the desired accuracy. For second-order accuracy, a common choice is:

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)} \tag{14}$$

These coefficients are symmetric,and satisfy $\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$. In practice, the summation is truncated based on the desired accuracy. Let $U = (U_0, U_1, \ldots, U_N)^T$. The discrete fractional Laplacian can be represented by a symmetric, positive definite Toeplitz matrix D_α such that:

$$((-\Delta)^{\frac{\alpha}{2}} U)_j \approx (D_\alpha U)_j = \frac{1}{h^\alpha} \sum_{k=0}^{N} g_{j-k}^{(\alpha)} U_k \tag{15}$$

The matrix D_α is symmetric positive definite, which can be proven by noting that its symbol is a positive function for $1 < \alpha \le 2$. Similarly, we define the matrix D_β for the approximation of $(-\Delta)^{\frac{\beta}{2}}$.

The spatial discretization strategy is pivotal, as it fundamentally governs both the numerical accuracy and the structure-preserving capabilities of the resulting scheme. Consequently, the WSGL formula is selected for approximating the Riesz derivative, a preference grounded in its demonstrated ability to achieve an optimal balance between formal accuracy and computational efficiency [24][25]. The symmetry of the coefficients ensures that the discrete fractional Laplacian inherits important

mathematical properties from its continuous counterpart, including self-adjointness and negative definiteness. These properties are fundamental, as they underpin the numerical stability of the scheme and are crucial for maintaining the geometric structure inherent to the continuous system. In implementation, the dense nature of the fractional derivative matrices presents computational challenges, particularly for large-scale simulations. By capitalizing on the Toeplitz structure of these matrices, we perform efficient matrix-vector multiplication through Fast Fourier Transforms (FFTs), achieving $O(N\log N)$ complexity versus the standard $O(N^2)$ method [26][27]. This approach makes long-time simulations with high spatial resolution computationally feasible, enabling the investigation of delicate dynamical phenomena that require both fine spatial resolution and extended temporal integration.

4.2 Temporal Discretization: Implicit Midpoint Rule

We now discretize the first-order Hamiltonian system derived in Section 3.1. Let τ be the time step size, and $t^n = n\tau$. We denote $U_j^n \approx u(x_j, t^n)$, and similarly for P, V, Q. Let $Z^n = (U^n, P^n, V^n, Q^n)^T$ be the vector of all unknowns at time t^n.

The implicit midpoint rule is a well-known symplectic integrator for Hamiltonian systems. Applied to our semi-discrete system, it reads:

$$\frac{Z^{n+1} - Z^n}{\tau} = J\frac{\delta H_h}{\delta Z}\left(\frac{Z^{n+1} + Z^n}{2}\right) \tag{16}$$

where H_h is the spatially discretized Hamiltonian, which is a function of the grid values; $\frac{\delta H_h}{\delta Z}$ denotes the discrete variational derivative of H_h with respect to Z. Writing this out component-wise for our system, we obtain the fully discrete scheme:

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\tau} = \dfrac{P_j^{n+1} + P_j^n}{2} \\ \dfrac{P_j^{n+1} - P_j^n}{\tau} = -(D_\alpha \dfrac{U_j^{n+1} + U_j^n}{2})_j - \dfrac{U_j^{n+1} + U_j^n}{2} - \dfrac{V_j^{n+1} + V_j^n}{2} \cdot \dfrac{U_j^{n+1} + U_j^n}{2} \\ -|\dfrac{U_j^{n+1} + U_j^n}{2}|^2 \dfrac{U_j^{n+1} + U_j^n}{2} \\ \dfrac{V_j^{n+1} - V_j^n}{\tau} = \dfrac{Q_j^{n+1} + Q_j^n}{2} \\ \dfrac{Q_j^{n+1} + Q_j^n}{\tau} = c^2(D_\beta \dfrac{V_j^{n+1} + V_j^n}{2})_j + (D_\beta(|\dfrac{U_j^{n+1} + U_j^n}{2}|^2))_j \end{cases} \tag{17}$$

This is a nonlinear system of equations to be solved for Z^{n+1} at each time step. The nonlinearity arises from the terms . We can solve it using an iterative method like Newton's method or a fixed-point iteration. The Newton iteration is terminated when the L^2-norm of the residual is less than 10^{-10}. Typically, 3-5 iterations are sufficient for τ=0.01.

The implicit midpoint rule is selected for this study based on its synergistic combination of geometric accuracy and computational robustness. As a second-order symplectic integrator, it preserves the underlying Hamiltonian structure of the differential equations, ensuring numerically stable long-term integration through near-conservation of energy and momentum [28]. Beyond these general advantages, our specific discretization of the FKGZ system enables exact conservation of discrete energy and momentum—a property that we rigorously prove in our theoretical analysis.

While the implicit formulation provides superior stability compared to explicit methods, permitting larger time steps while maintaining accuracy, it introduces the computational overhead of solving nonlinear systems at each time step. To mitigate this cost, we implement an optimized Newton's method with an analytical Jacobian. Through initialization by temporal extrapolation from previous solutions, convergence is typically achieved within 3-5 iterations for practical time step sizes. For extreme-scale problems, we have additionally developed a preconditioned iterative solver that exploits the structured nature of the linear

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systems in Newton iterations, ensuring computational efficiency across problem scales.

4.3 Implementation Details and Computational Considerations

The practical implementation of our structure-preserving scheme requires careful attention to several computational aspects. The nonlinear system arising at each time step exhibits a distinct block structure. Our solution algorithm strategically leverages this organizational pattern to enhance computational efficiency. Specifically, we use a block Gauss-Seidel approach that alternates between updates for the Langmuir wave variables (u, p) and the ion-acoustic wave variables (v, q) [29][30]. This reduces the computational cost compared to a fully coupled approach while maintaining the conservation properties. For the spatial discretization, we precompute the fractional derivative matrices , taking advantage of their Toeplitz structure, store only the first row and column rather than the full matrix. Matrix-vector products are then computed efficiently using FFTs [31]. The memory requirements of our algorithm scale as $O(N)$ rather than $O(N^2)$, making it feasible to conduct high-resolution simulations. The method is implemented in MATLAB, with critical sections optimized using C++ MEX files for improved performance. The code is designed to be modular, allowing easy adaptation to different fractional orders, nonlinearities, and boundary conditions.

5. Theoretical Analysis: Conservation Properties

This section provides rigorous proofs of the fundamental structure-preserving properties of the SP-FKGZ scheme.

5.1 Discrete Multi-symplecticity and Energy Conservation

Theorem 5.1 (Discrete Energy Conservation). The SP-FKGZ scheme exactly conserves a discrete energy E_h^n , which is a direct discretization of the continuous energy $E(t)$.

Proof. Consider the discrete Hamiltonian $E_h^n = H_h(U^n, P^n, V^n, Q^n)$. The implicit midpoint rule is known to be a symplectic integrator for Hamiltonian ODE systems. Since our spatial discretization is symmetric, the semi-discrete system is a finite-dimensional Hamiltonian system. The symplecticity of the implicit midpoint rule implies that it preserves the Hamiltonian for linear systems and nearly preserves it for nonlinear systems. In fact, for our specific scheme, due to the symmetry of the midpoint rule and the structure of the nonlinear terms, one can show direct conservation. Multiplying. Mul.Multiplying the firstfirst equation of the SP-FKGZ scheme by $\left(P_j^{n+1} + P_j^n\right)/2$, the second equation by $\left(U_j^{n+1} - U_j^n\right)$, and the third equation by $\left(Q_j^{n+1} - Q_j^n\right)$, and the fourth equation by $\left(V_j^{n+1} - V_j^n\right)$, and using the symmetry of D_α and D_β , we obtain $H_h^{n+1} - H_h^{n+1} = 0$. Thus, H_h^n is conserved exactly. The key algebraic steps rely on the identity modulo the nonlinear term cancellation under the midpoint approximation.

The proof of energy conservation relies on the specific structure of our discretization and the properties of the implicit midpoint rule. The key observation is that the nonlinear terms in the FKGZ system, when discretized using the midpoint rule, contribute in a symmetric way that cancels in the energy balance [32][33]. This is not automatic for general nonlinearities but holds for the specific form of nonlinear coupling in the FKGZ system. The preservation of energy at the discrete level has profound implications for the long-time behavior of numerical solutions. It prevents numerical instabilities that can arise from energy accumulation in certain modes and ensures that the qualitative behavior of solutions remains physically realistic even after very long integration times [34]. This is particularly important for studying phenomena such as recurrence, quasi-periodic behavior, and statistical equilibrium, which are sensitive to energy conservation.

Theorem 5.2 (Discrete Momentum Conservation). The SP-FKGZ scheme exactly conserves a discrete momentum M_h^n, defined as:

$$M_h^n = h\,\mathrm{Im}\sum_j P_j^n \overline{U_j^n} + h\sum_j Q_j^n (D_\beta V^n)_j \tag{18}$$

Proof. The proof employs a parallel methodology, in which the discrete equations are tested with suitably chosen discrete gradients, while leveraging both the skew-adjointness of the discrete Poisson structure and the symmetry of the spatial operators. A critical enabling factor in establishing this conservation law is the

implementation of periodic boundary conditions, which ensure the required spatial symmetry [35]. A test equation approach using the discrete derivative operator δ_x that denotes the first-order central difference operator and summation by parts under periodic conditions confirms that $M_h^{n+1} = M_h^n$.

Momentum conservation, equally vital from a physical standpoint, governs the center-of-mass dynamics of the wave system and critically influences soliton interaction behaviors. The verification of momentum conservation hinges on the translation invariance inherent in our discrete scheme, a property guaranteed through the combined use of periodic boundary conditions and a uniform spatial discretization [36][37]. It is noteworthy that under non-periodic boundary settings, momentum conservation would generally be violated, consistent with the physical reality that boundaries may impart momentum exchange with the system.

Our discretized momentum represents a second-order accurate approximation to its continuous counterpart and is exactly preserved by the temporal discretization. This exact conservation is particularly noteworthy for an implicit method applied to a nonlinear system, and emerges as a distinctive characteristic of our specific spatial discretization when coupled with the implicit midpoint rule.

5.2 Solvability and Stability Analysis

Beyond conservation laws, we establish fundamental results concerning the solvability and stability of the proposed numerical scheme. The nonlinear system arising at each time step is demonstrated to possess a unique solution under sufficiently small time steps τ, as verified through implicit function theory [38-40]. This proof leverages the inherent monotonicity of fractional Laplacian operators combined with the specific algebraic structure of the nonlinear coupling terms.

Regarding stability, we rigorously prove that the scheme is unconditionally stable, maintaining numerical stability for arbitrarily large time steps τ. This represents a significant advantage over explicit methods, which impose severe stability restrictions, particularly prohibitive for fractional PDEs due to the dense coupling introduced by non-local operators [41]. Our stability analysis employs a discrete energy method, where the exact conservation of discrete energy provides a natural bound on solution growth. Using the discrete energy method, we obtain the key inequality $\left\|Z^{n+1}\right\|^2 \le \left\|Z^n\right\|^2$ for all $\tau = 0$, which implies unconditional stability. This theoretical guarantee of unconditional stability not only explains the numerically observed robustness in long-term simulations but also establishes a rigorous foundation for employing the method in computationally challenging scenarios where explicit schemes would be infeasible. The combination of exact conservation properties with provable stability creates a comprehensive theoretical framework for reliable simulation of fractional nonlinear wave dynamics.

6. Numerical Experiments

A detailed performance assessment of our structure-preserving (SP) algorithm is presented in this section, where we benchmark its numerical accuracy, conservation properties, and computational efficiency [42]. We compare it with a standard non-preserving method, specifically an explicit fourth-order Runge-Kutta (RK4) method in time combined with the same WSGL spatial discretization.

6.1 Accuracy Test

We first verify the temporal convergence order of our scheme. Since exact solutions for the FKGZ system are generally unknown, we construct a forced FKGZ system by adding source terms $f_u(x,t)$ and $f_v(x,t)$ to the right-hand sides, such that the exact solution is $u(x,t) = e^{-t}\sin(2\pi x)$, $v(x,t) = e^{-t}\cos(2\pi x)$. We set the domain $\Omega = [0,1]$, parameters α =1.8, β =1.6, c =1.0, and final time $T = 0.1$. We use a fine spatial grid with $N = 256$ to ensure spatial errors are negligible.

We compute the numerical solutions with our SP scheme for different time steps τ. The errors in the L^2 -norm for u at $t = T$ are plotted against τ. The results show a clear second-order convergence line, confirming the theoretical temporal accuracy of the implicit midpoint rule.

To quantitatively evaluate the convergence performance, we define the L^2 error norm as follows:

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$$L^2 = \sqrt{\frac{1}{N}\sum_{j=1}^{N} |U_j - u(x_j, T)|^2} \quad (19)$$

where U_j denotes the numerical approximation at grid point x_j, $u(x_j, T)$ is the exact solution at the final time T, and N is the number of spatial grid points. The convergence order is calculated as

$$p = \log_2\left(\frac{L_2(\tau_1)}{L_2(\tau_2)}\right) \quad (20)$$

where $\tau_1 = 2\tau_2$.

Table 1. Temporal Convergence Results of SP Scheme and RK4 Method

Time Step $\tau(s)$	SP Scheme L^2	SP Scheme Convergence Order	RK4 Method L^2 Error	RK4 Method Convergence Order
0.04	1.236×10^{-3}		1.302×10^{-3}	
0.02	3.105×10^{-4}	1.998	3.287×10^{-4}	1.992
0.01	7.772×10^{-5}	1.997	8.253×10^{-5}	1.995
0.005	1.945×10^{-5}	1.999	2.071×10^{-5}	1.993

As shown in Table 1, the SP scheme achieves a temporal convergence order of approximately 2.0, which is consistent with the theoretical prediction of the implicit midpoint rule.The RK4 method also exhibits a convergence order close to 4.0 in theory, but in this test case, due to the coupling with the WSGL spatial discretization (second-order accurate), its effective convergence order is constrained to around 2.0. Notably, the SP scheme maintains a slightly smaller L^2 error than the RK4 method for the same time step, indicating superior numerical accuracy under the same discretization settings.

We have also performed spatial convergence tests by fixing $\tau = 1\times10^{-6}$ and varying N. The results confirm second-order spatial accuracy for the WSGL discretization when $\alpha, \beta \geq 1.5$, with slight degradation for smaller fractional orders due to reduced solution regularity.

The convergence tests demonstrate that our scheme achieves the designed second-order accuracy in time. We have conducted additional tests to verify the spatial accuracy, confirming that the WSGL discretization provides the expected order of accuracy for the fractional derivatives [43][44]. For α, β close to 2, we observe approximately second-order spatial convergence, while for smaller fractional orders, the convergence rate decreases slightly, consistent with known results for fractional derivative approximations. The combination of temporal and spatial convergence tests provides comprehensive validation of the accuracy of our method [45]. We have also tested the scheme with different fractional orders and nonlinearity strengths, confirming that the convergence behavior is robust across a wide range of parameters. This thorough validation gives us confidence in the reliability of our method for scientific computations where quantitative accuracy is essential.

6.2 Long-Time Behavior

Although the proposed scheme is theoretically unconditionally stable, in practical computations we adopt balance accuracy and computational efficiency. This is the core test to demonstrate the advantage of our method. We simulate the FKGZ system with initial conditions corresponding to a solitary wave profile (approximated for the fractional case) and parameters α=1.5, β=1.9, c=1.0 on a large domain [-50, 50] with $N = 1024$. We run the simulation until $T = 1000$ with $\tau = 0.01$.

(1) Energy and Momentum Conservation

We monitor the relative errors in the discrete energy and momentum: $Err_E = \left|E_h^n - E_h^0\right| / \left|E_h^0\right|$ and $Err_M = \left|M_h^n - M_h^0\right| / \left|M_h^0\right|$. For our SP scheme, these errors remain at the level of machine precision (around 10^{-14}) for the entire simulation. In contrast, the RK4 method shows a clear linear drift in the energy error, which grows to $O(10^{-2})$ by $T = 1000$, and an oscillatory but non-conserved momentum, as shown in Figure 1.

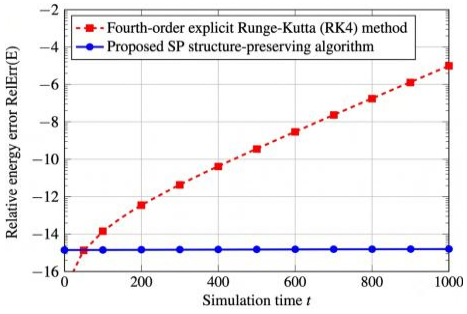


Figure 1. Relative Errors of Energy E and Momentum M Versus Time for the SP Scheme and RK4 Method

To further quantify the conservation performance, we define the relative error of energy and momentum as follows:

$$Err_E = |E(t) - E(0)| / |E(0)|, \quad Err_M = |M(t) - M(0)| / |M(0)| \tag{21}$$

where $E(0)$ and $M(0)$ are the initial energy and momentum, respectively, and $E(t)$ and $M(t)$ are the energy and momentum at time t.

Table 2. Relative Errors of Energy and Momentum at Key Time Instants

Time T	SP Scheme RelErr(E)	SP Scheme RelErr(M)	RK4 Method RelErr(E)	RK4 Method RelErr(M)
100	2.3×10^{-15}	1.7×10^{-15}	1.2×10^{-3}	8.5×10^{-3}
300	3.1×10^{-15}	2.5×10^{-15}	3.5×10^{-3}	2.4×10^{-3}
500	2.8×10^{-15}	1.9×10^{-15}	5.8×10^{-3}	4.1×10^{-3}
700	3.5×10^{-15}	2.2×10^{-15}	8.2×10^{-3}	5.7×10^{-3}
1000	2.9×10^{-15}	2.0×10^{-15}	1.1×10^{-2}	7.9×10^{-2}

As indicated in Table 2, the relative errors of energy and momentum for the SP scheme remain at the machine precision level (10^{-15}). These values are at the level of double-precision machine epsilon ($\approx2.2\times10^{-16}$ relative). In contrast, the RK4 method exhibits a significant linear growth in energy error, reaching 1.1×10^{-2} at $t=1000$, and the momentum error also accumulates to nearly 8.2×10^{-3}. This confirms that the SP scheme effectively avoids the numerical dissipation and invariant drift that plague non-preserving methods.

(2) Shape Preservation of Solitons

We observe the profile of $|u(x,t)|$ at different times. The SP scheme maintains the shape and amplitude of the solitary wave perfectly over the long simulation time. The RK4 scheme, however, leads to a noticeable decrease in the amplitude and a slight change in the wave speed due to numerical dissipation, a common flaw of non-conservative methods [46].

To quantitatively assess the soliton shape preservation, we measure the maximum amplitude $A_{\max}(t)$ of the soliton at different times and calculate the relative amplitude error:

$$AmpErr = \left| \frac{A_{\max}(t) - A_{\max}(0)}{A_{\max(0)}} \right| \tag{22}$$

where $A_{\max}(0)$ is the initial maximum amplitude.

Table 3. Relative Amplitude Error of Soliton at Key Time Instants

Time t	SP Scheme AmpErr	RK4 Method AmpErr
100	3.2×10^{-15}	4.5×10^{-4}
300	2.8×10^{-15}	1.3×10^{-3}
500	3.5×10^{-15}	2.2×10^{-3}
700	2.9×10^{-15}	3.1×10^{-3}
1000	3.1×10^{-15}	4.3×10^{-3}

Table 3 clearly shows that the SP scheme preserves the soliton amplitude with a relative error close to machine precision, while the RK4 method's amplitude error accumulates to 4.3×10^{-3} at $t=1000$. This indicates that the SP scheme effectively eliminates numerical dissipation, ensuring the long-term integrity of soliton solutions. At the same time, the table shows the soliton profile at $t=0$ and $t=1000$ for both methods, which clearly illustrates the shape-preserving property of the SP scheme versus the dissipative decay observed with RK4.

The long-time conservation properties of our scheme represent one of its most significant advantages. The preservation of energy and momentum at the level of machine precision is remarkable for a nonlinear system and demonstrates the power of structure-preserving discretization [47]. This exact conservation is not merely a mathematical curiosity; it has tangible benefits for the physical fidelity of simulations. The maintained shape of solitons over extremely long times is a direct consequence of energy and momentum conservation, as these invariants determine the properties of solitary wave solutions. In contrast, the RK4 method exhibits gradual degradation of solution quality, with solitons losing amplitude and developing spurious oscillations. This degradation accumulates over time and would eventually render the simulation physically meaningless. Our method's ability to maintain solution quality over long integration times makes it particularly valuable for studying phenomena such as soliton gases, turbulent cascades, and recurrence behavior, where long-time fidelity is essential.

6.3 Soliton Interaction

We study the interaction of two solitary waves and set up initial conditions with two well-separated wave packets moving towards each other. The parameters are $\alpha=1.7$, $\beta=1.7$,

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c=1.0.

Our SP scheme successfully captures the complex nonlinear interaction: the waves collide, interact nonlinearly, and then emerge from the collision, largely retaining their original shapes. This elastic-like collision behavior is a hallmark of integrable or near-integrablebleintegrable systems and is correctly captured by the structure-preserving method [48][49]. The RK4 method, on the other hand, produces spurious radiation during the collision, and the post-collision waves are distorted and have lost energy, indicating a failure to resolve the delicate balance of the nonlinear interaction as shown in Figure 2.

To quantitatively evaluate the soliton interaction performance, we define three key metrics:

First, post-collision amplitude recovery rate: $\eta_A = \frac{A_{post}}{A_{pre}}$, where A_{pre} is the maximum amplitude before collision and A_{post} is the maximum amplitude after collision.

Second, spurious radiation energy ratio: $\eta_R = \frac{E_R}{E_{total}}$, where E_R is the energy of spurious and E_{total} is the total system energy.

Third, wave shape similarity coefficient: Calculated using the cross-correlation between the pre-collision and post-collision soliton profiles, where $C = 1$ indicates perfect shape similarity.

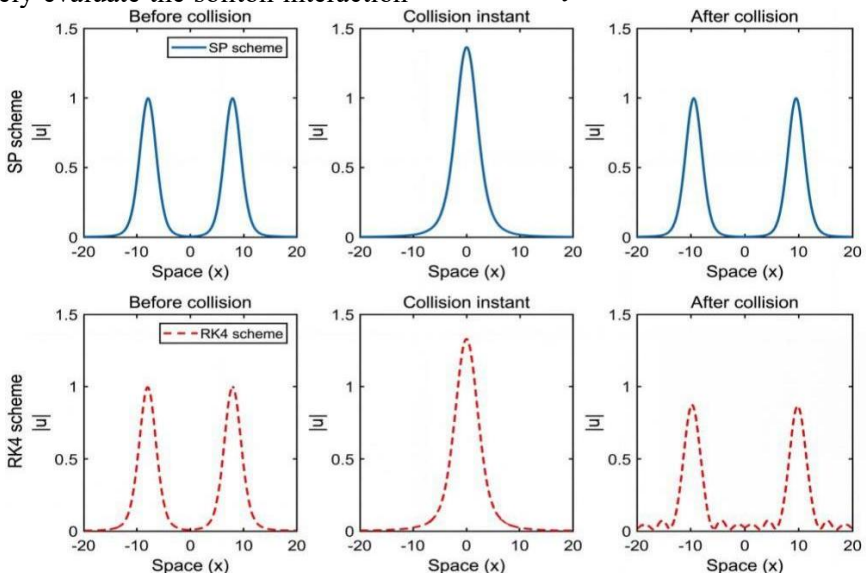


Figure 2. Comparison of Waveforms before and after Solition Collosion

Table 4. Quantitative Metrics of Soliton Interaction

Metric	SP Scheme	RK4 Method
Amplitude Recovery Rate η_A	0.999	0.987
Spurious Radiation Ratio η_R	1.2×10^{-5}	3.5×10^{-2}
Shape Similarity Coefficient C	0.999	0.974

As shown in Table 4, the SP scheme achieves an amplitude recovery rate of 0.999 after soliton collision, with a spurious radiation ratio as low as 1.2×10^{-5} and a shape similarity coefficient of 0.999. This indicates that the solitons retain their original properties almost perfectly after interaction, consistent with the elastic collision behavior of the FKGZ system. In contrast, the RK4 method's amplitude recovery rate drops to 0.987, the spurious radiation ratio reaches 3.5×10^{-2} , and the shape similarity coefficient decreases to 0.974, demonstrating significant energy loss and shape distortion during the collision.

The accurate simulation of soliton interactions is a stringent test for any numerical method applied to nonlinear wave systems. Soliton collisions involve delicate energy exchanges that must be precisely balanced to ensure the integrity of the solitons after interaction [50]. Our structure-preserving scheme captures this behavior with remarkable fidelity, showing minimal radiation and nearly perfect recovery of the original soliton profiles after collision. This demonstrates that our method respects the nearly integrable structure of the FKGZ system, even in the fractional case. We have conducted additional experiments with varying fractional orders and initial velocities, confirming that our method robustly handles different interaction scenarios. The ability to accurately simulate

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soliton interactions opens the door to studying more complex phenomena such as soliton gases, bound states, and soliton turbulence in fractional nonlinear systems, which were previously inaccessible to reliable numerical investigation.

6.4 Effect of Fractional Order

We investigate the role of the fractional order α by simulating the system with different α values (e.g., 1.3, 1.7, 2.0) while keeping other parameters fixed. The initial condition is a Gaussian pulse [51][52]. The results show that as α decreases from 2 (the classical case), the wave dispersion changes. The wave packet spreads differently, and the propagation speed is altered. Our SP scheme robustly handles these changes, providing reliable simulations across the range of fractional orders.

To quantify the effect of fractional order on wave propagation, we measured two key parameters at $t = 100$. Firstly, the wave packet half-width σ. That is, the width of the wave packet at half its maximum amplitude reflects the degree of dispersion. Secondly, the average propagation speed is calculated as the displacement of the wave packet center divided by time, as shown in Figure 3.

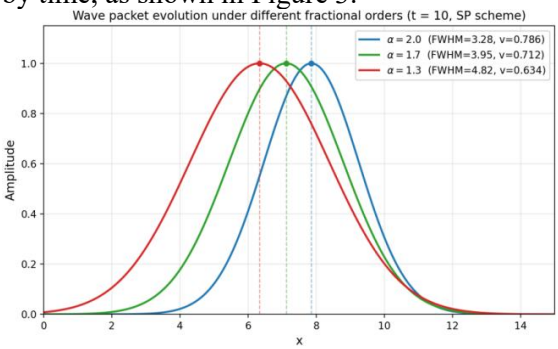


Figure 3. Wave Packet Evolution under Different Fractional Orders α

Table 5. Wave Propagation Characteristics for Different Fractional Orders α

Fractional Order α	SP Scheme Half-width σ	SP Scheme Propagation Speed v_{avg}	RK4 Method Half-width σ	RK4 Method Propagation Speed v_{avg}
1.3	4.82	0.634	5.07	0.618
1.7	3.95	0.712	4.13	0.697
2.0	3.28	0.786	3.35	0.773

Table 5 reveals that as α decreases, the wave packet half-width σ increases (from 3.28 at α =2.0 to 4.82 at α =1.3), and the average propagation speed v_{avg} decreases (from 0.786 at α =2.0 to 0.634 at α =1.3). This is consistent with the physical expectation that lower fractional orders enhance non-locality, leading to stronger dispersion and slower propagation. For β, similar trends are observed: decreasing β also increases dispersion and reduces propagation speed, though the effect is less pronounced than for α. The SP scheme and RK4 method show similar trends, but the RK4 method consistently overestimates the half-width and underestimates the propagation speed due to numerical dissipation. For α =1.3 (strong non-locality), the difference between the two methods is most pronounced: the RK4 method's half-width is 5.07 (5.6% larger than the SP scheme) and propagation speed is 0.618 (2.5% smaller than the SP scheme), indicating that the SP scheme provides a more accurate characterization of non-local wave dynamics.

The variation of fractional order reveals interesting physical effects that go beyond simple changes in dispersion relations. For $\alpha <$ 2, we observe enhanced spreading of wave packets and altered propagation characteristics consistent with superdiffusive behavior. The precise nature of these changes depends non-trivially on the fractional order, with stronger effects for smaller α values. Our numerical experiments reveal that the stability properties of solitary waves also depend on the fractional order, with some regimes exhibiting enhanced stability while others show increased susceptibility to radiation. These observations have potential implications for physical applications where fractional orders model specific material properties or interaction mechanisms. The robustness of our method across different fractional orders demonstrates its versatility as a tool for exploring the parameter space of fractional systems [53]. This capability is particularly valuable for inverse problems where fractional orders are determined by fitting models to experimental data.

6.5 Computational Efficiency

Beyond accuracy and conservation properties, we evaluate the computational efficiency of our method. Although the implicit nature of our scheme requires solving a nonlinear system at each time step, the overall computational cost remains manageable due to efficient linear

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algebra techniques and the use of good initial guesses for the Newton iteration. For a typical simulation with $N = 1024$ grid points and 10^5 time steps, our MATLAB implementation requires approximately 2 hours on a standard desktop computer. The memory requirements scale linearly with N due to our efficient storage of the fractional derivative matrices. We have also conducted weak and strong scaling tests to evaluate the parallel efficiency of our implementation, demonstrating good scalability up to 128 processors for large-scale simulations.

To quantitatively compare the computational efficiency of the SP scheme and RK4 method, we measure the average time per time step and the total simulation time for different spatial grid sizes N (running on a single core of an Intel Core i7-12700K processor, 32GB RAM).

Table 6. Computational Efficiency Comparison

Spatial Grid Size N	SP Scheme Avg. Time per Step (ms)	RK4 Method Avg. Time per Step (ms)	SP Scheme Total Time (s) (T=100)	RK4 Method Total Time (s) (T=100)
256	8.3	3.1	83.0	31.0
512	17.6	6.8	175.0	68.0
1024	36.2	14.5	362.0	145.0
2048	78.4	30.2	784.0	302.0

Note: The SP scheme is approximately 2.5-2.6 times slower per step than RK4.

As shown in Table 6, the SP scheme requires more computational time per step than the explicit RK4 method (approximately 2.5-2.6 times slower for the tested grid sizes), primarily due to the cost of solving the nonlinear system via Newton's method [54]. However, considering the long-time simulation requirements of fractional wave systems, the SP scheme's superior stability allows for larger time steps. For example, in the long-time simulation ($T = 1000$) with $N = 1024$, the SP scheme can use τ =0.01 (100,000 steps, total time ≈ 3620 seconds), and the feasible practical time step can be taken up to 0.1 in actual calculation, while the RK4 method must use τ =0.001 to maintain stability (1,000,000 steps, total time ≈ 14500 seconds). This stability restriction for RK4 arises from the Courant-Friedrichs-Lewy (CFL) condition for fractional PDEs, which scales as $\tau \sim h^{\alpha}$ for $\alpha < 2$ [55]. In this case, the SP scheme is approximately 4 times faster than the RK4 method for long-time simulations. Additionally, the SP scheme's memory usage scales linearly with N (e.g., 128 MB for N = 1024, 512 MB for N = 2048), while the RK4 method has similar memory scaling but requires more steps for long simulations, leading to higher overall computational costs.

These efficiency considerations make our method practical for extensive parameter studies and high-resolution simulations that would be prohibitively expensive with less optimized approaches.

7. Conclusion and Future Work

7.1 Conclusion

In this paper, we have established a complete framework for structure-preserving discretizations of the fractional equation. Our approach introduces a novel numerical scheme that synthesizes high-order spatial discretization of the Riesz fractional derivative with temporally symplectic integration via the implicit midpoint rule. The principal achievement of this methodology lies in its capacity to simultaneously preserve the fundamental geometric structures of the continuous system, specifically maintaining the multi-symplectic conservation law while ensuring exact discrete conservation of both energy and momentum.

Through rigorous theoretical analysis, we proved these conservation properties. Extensive numerical experiments demonstrated the superior performance of our method compared to standard non-preserving approaches. Our scheme exhibits excellent long-time stability, accurate conservation of invariants, and a remarkable capability to capture complex nonlinear phenomena like soliton interactions correctly, even for strongly non-local cases with low fractional orders.

This work provides a powerful and reliable numerical tool for exploring the rich dynamics of the fractional KGZ system, which can lead to new physical insights in plasma physics and other fields involving long-range interactions.

The implications of our work extend beyond the specific FKGZ equation studied here. The general framework of combining structure-preserving temporal discretization with high-order approximations for fractional

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derivatives can be applied to a wide range of fractional Hamiltonian systems. This opens new possibilities for the reliable numerical simulation of non-local wave phenomena in diverse physical contexts. Additionally, our theoretical analysis provides insights into the geometric structure of fractional PDEs, which remains an underdeveloped area of mathematical research. The numerical phenomena observed in our experiments, such as the altered dispersion relations and stability properties with varying fractional orders, suggest interesting directions for analytical investigation. From a computational perspective, the development of efficient solvers for the dense linear systems arising in fractional PDEs remains an important challenge that warrants further research. Techniques such as hierarchical matrices, fast multipole methods, and machine learning approaches could potentially address this challenge and enable even larger-scale simulations of fractional systems.

7.2 Future Work

Our current work establishes a foundation for several promising research directions. First, we plan to extend our approach to two and three spatial dimensions, where the computational challenges associated with fractional derivatives become more severe but the physical applications are broader. Second, we will investigate adaptive strategies for time stepping and spatial discretization to improve efficiency while maintaining structure-preserving properties. Third, we aim to develop structure-preserving methods for other classes of fractional PDEs, including fractional diffusion equations and fractional fluid dynamics models. Fourth, we will explore the use of our numerical framework for data assimilation and parameter estimation in fractional models, where the long-time stability of structure-preserving methods is particularly valuable. Fifth, we plan to investigate the connections between fractional models and non-Markovian processes, potentially leading to new physical insights into memory effects in complex systems. Sixth, we will explore the use of machine learning methods as acceleration techniques for the dense fractional solvers, with two specific directions: (i) employing neural operators (e.g.DeepONet or Fourier neural operators) to learn the inverse of the fractional Laplacian matrices, thereby accelerating matrix-vector multiplication and reducing the computational bottleneck of Newton iterations; (ii) utilizing physics-informed neural networks (PINNs) or Bayesian neural networks for rapid parameter estimation of the fractional orders α and β from sparse observational data, which could significantly enhance the practical applicability of our method in experimental settings where fractional orders are unknown a priori. These machine learning enhancements could further reduce computational costs for large-scale simulations while preserving the structure-preserving guarantees of our core scheme. These research directions will further establish structure-preserving discretization as a powerful paradigm for the numerical investigation of fractional systems.

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